Categorical semantics for model comparison games for description logics

(Semantyka kategoryczna dla gier porównujących modele
dla logik deskrypcyjnych)

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Abstract

A categorical approach to study model comparison games in terms of comonads was recently initiated by Abramsky et al. In this work, we analyse games that appear naturally in the context of description logics and supplement them with suitable game comonads. More precisely, we consider expressive sublogics of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$, namely, the logics that extend $\mathcal{ALC}$ with any combination of inverses, nominals, safe boolean roles combinations, and Self operator. Our construction augments and modifies the so-called modal comonad by Abramsky and Shah. The approach that we took heavily relies on the use of relative comonads, which we leverage to encapsulate additional capabilities within the bisimulation games in a compositional manner.

Kategoryczne podejście do badania gier porównujących modele za pomocą komonad zostało niedawno zainicjowane przez Abramsky'iego et al. W tej pracy, analizujemy gry które pojawiają się naturalnie w kontekście logik deksrypcyjnych i uzupełniamy je o odpowiednie komonady gier. Mówiąc dokładniej, rozważamy ekspresywne podlogiki $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$, mianowicie, logiki które rozszerzają $\mathcal{ALC}$ o inwersje, stałe, bezpieczne kombinacje boolowskie ról oraz operator Self. Nasza konstrukcja modyfikuje tak zwaną modalną komonady od Abramsky'iego i Shah'a. Podejście które przyjęliśmy opiera się w głównej mierze na użyciu relatywnych komonad, które wykorzystujemy aby wyrazić dodatkowe możliwości w grach bisymulacyjnych w sposób, który się dobrze składa.

Contents

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Chapter 1

Introduction

Following [2], there are two different views on the fundamental features of computation, that can be summarised as "structure" and "power" as follows:

- **Structure**: Compositionality and semantics, addressing the question of mastering the complexity of computer systems and taming computational effects.
- **Power**: Expressiveness and complexity, addressing the question of how we can harness the power of computation and recognize its limits.

It turned out that there are almost disjoint communities of researchers studying Structure and Power, with seemingly no common technical language and tools. To encounter this issue, Samson Abramsky and Anuj Dawar started a project, whose goal is to provide a category-theoretical toolkit to reason about finite model theory in order to apply theorems and draw insights from, at first sight, an unrelated field.

Their approach employs comonads on the category of relational structures to capture denotational semantics of model comparison games such as Ehrenfeucht-Fraïssé, pebbling, and bisimulation games [8], as well as games for Hybrid logics [6] and Guarded Fragment [5]. The structure allows us to leverage the tool of category theory, and apply it to generalise known established theorems, as it was done in [13] or [7]. In this paper, we continue the exploration of suitable game comonads by incorporating the comonadic semantics for description logics games, namely, for expressive description logics between $\mathcal{ALC}$ and $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$. [1] It is also worth mentioning parallel research that defines categorical semantics for $\mathcal{ALC}$ [15, 12], however, their approach is much different from ours, as we focus solely on games and leave $\mathcal{ALC}$ in the standard set-theoretic semantics.

[1] It will become clear why we write $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$ instead of the more standard form $\mathcal{ALCOI}b_{\mathsf{Self}}$ later.

1.1 Our results

In what follows, we change the setting established in the previous work [8] from the category of relational structures to a category of pointed interpretations that are parametrised by sets of role names, concept names and individual names.

We start by defining comonadic semantics for $\mathcal{ALC}$-bisimulation-games. It is well-known that $\mathcal{ALC}$ is a notational variant of multi-modal logic. Hence, we employ this observation to take full advantage of existing results on modal logic from [8] and use them as the base for our further investigations. In order to define comonadic semantics for DLs $\mathcal{L}\Phi \subseteq \mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$, instead of providing it directly for them (and thus repeating all the machinery and required proofs from [8]), we follow a different route. We provide a family of game reductions from $\mathcal{L}\Phi$ to weaker sublogics, ending up on $\mathcal{ALC}$, which transform interpretations in such a way that a winning strategy in $\mathcal{L}\Phi$-bisimulation-game is equivalent to a winning strategy in $\mathcal{ALC}$-bisimulation-game for suitably transformed interpretations. From a categorical point of view, we introduce a comonad for $\mathcal{ALC}$ logic and reductions shall be defined by functors, on which we will build relative comonads to encapsulate the additional capabilities available in an $\mathcal{L}$-bisimulation-game. By composing the reduction functors together, we shall obtain comonadic semantics for all of the games for considered logics.

1.2 Roadmap

We start in Chapter 2 by giving a sufficient background for the further results, to facilitate the accessibility for readers coming both from the area of model theory and description logics, as well as from the category theory side.

In Chapter 3, we recall the well-established notion of bisimulation games for $\mathcal{L} \subseteq \mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$ logics, which are the key concept for which we shall define the categorical semantics.

We then proceed to Chapter 4, where we define a family of logic extension reductions $\mathfrak{f}_{\mathsf{Self}}, \mathfrak{f}_{\mathcal{I}}, \mathfrak{f}_b, and \mathfrak{f}_{\mathcal{O}}$ acting on interpretations. We declare them with a goal such that for $\Phi \subseteq \{\mathsf{Self}, \mathcal{I}, b, \mathcal{O}\}$ and $\mathfrak{f}_\Phi$ being a composition of reductions of extensions selected by Φ, the following theorem holds:

$$(\mathcal{I}, \mathrm{d}) \sim_k^{\mathcal{ALC}\Phi} (\mathcal{J}, \mathrm{e}) \iff (\mathfrak{f}_\Phi\, \mathcal{I}, \mathrm{d}) \sim_k^{\mathcal{ALC}} (\mathfrak{f}_\Phi\, \mathcal{J}, \mathrm{e})$$

Having established model-theoretic part of our work, we finally move to the category theory world, where we shall stay until for the rest of the thesis. Chapter 5 tweaks modal comonad and ports categorical variation of comparison games from [8] such that it can be applied to our description logic setting. We wrap up the chapter by giving denotational, comonadic semantics for $\mathcal{ALC}$-bisimulation-games.

Finally, in Chapter 6, we devise a general framework for establishing comonadic semantics for games for all expressive sublogics of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$. We achieve this by lifting previously defined reductions to well-behaved functors and taking a relative comonad over them.

We conclude in Chapter 7 by suggesting potential future research directions as well as giving motivation to the thesis by presenting what was already achieved in this field by leveraging the developed toolkit.

Chapter 2

Preliminaries

We start with a recap of notions from category theory [10, 17], such as comonads, as well as from description logics, for which we define their syntax, semantics and bisimulations [11]. By doing so, we would like to unify the context for readers from different backgrounds.

2.1 Preliminaries on DLs.

We fix infinite mutually disjoint sets of *individual names* $\mathbf{N_I}$, *concept names* $\mathbf{N_C}$, and *role names* $\mathbf{N_R}$. We will briefly recap the syntax and semantics of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$-concepts and as well as $\mathcal{L}$-concepts for relevant sublogics $\mathcal{L}$ of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$. The following EBNF grammar defines *atomic concepts* B, *concepts* C, *atomic roles* r, *simple roles* s with $\mathrm{o} \in \mathbf{N_I}$, $\mathrm{A} \in \mathbf{N_C}$, $p \in \mathbf{N_R}$:

$$\begin{array}{rcl} \mathrm{B} & ::= & \mathrm{A} \mid \{\mathrm{o}\} \\ \mathrm{C} & ::= & \mathrm{B} \mid \neg\mathrm{C} \mid \mathrm{C} \sqcap \mathrm{C} \mid \exists s.\mathrm{C} \mid \exists s.\mathsf{Self} \\ r & ::= & p \mid p^- \\ s & ::= & r \mid s \cap s \mid s \cup s \mid s \setminus s \end{array}$$

The semantics of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$-concepts is defined via *interpretations* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ composed of a non-empty set $\Delta^{\mathcal{I}}$ called the *domain of* $\mathcal{I}$ and an *interpretation function* $\cdot^{\mathcal{I}}$ mapping individual names to elements of $\Delta^{\mathcal{I}}$, concept names to subsets of $\Delta^{\mathcal{I}}$, and role names to subsets of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. This mapping is then extended to complex concepts and roles (*cf.* Table 2.1). The *rank* of a concept is the maximal nesting depth of $\exists$-restrictions.

We shall use expressions of the form $\mathcal{ALC}\Phi$ or $\mathcal{L}\Phi$ with $\Phi \subseteq \{\mathcal{O}, \mathcal{I}, \mathsf{Self}, b\}$ to speak collectively about different expressive sublogics of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$.

The $\mathcal{ALC}\Phi$-concepts are obtained by dropping from the syntax the inversions of roles ($\mathcal{I}$), safe boolean combination of roles (b) (*i.e.* role union, intersection and difference), nominals ($\mathcal{O}$) and the self operator (Self), depending on the content of Φ. We stress here that role union/intersection/difference, the Self operator, role inverse $\cdot^{-}$ and nominals $\{\cdot\}$ are just operators and they introduce neither new role names nor new concept names.

Name	Syntax	Semantics
concept name	A	$\mathrm{A}^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
role name	r	$r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
concept negation	$\neg\mathrm{C}$	$\Delta^{\mathcal{I}} \setminus \mathrm{C}^{\mathcal{I}}$
concept intersection	$\mathrm{C} \sqcap \mathrm{D}$	$\mathrm{C}^{\mathcal{I}} \cap \mathrm{D}^{\mathcal{I}}$
existential restriction	$\exists r.\mathrm{C}$	$\{\, \mathrm{d} \mid \exists \mathrm{e}.(\mathrm{d}, \mathrm{e}) \in r^{\mathcal{I}} \wedge \mathrm{e} \in \mathrm{C}^{\mathcal{I}} \,\}$
nominal op.	$\{\mathrm{o}\}$	$\{\mathrm{o}^{\mathcal{I}}\}$
inverse role op.	p^{-}	$\{(\mathrm{d}, \mathrm{e}) \mid (\mathrm{e}, \mathrm{d}) \in p^{\mathcal{I}}\}$
role boolean op. for $\oplus \in \{\cup, \cap, \setminus\}$	$s_1 \oplus s_2$	$s_1^{\mathcal{I}} \oplus s_2^{\mathcal{I}}$
Self op.	$\exists s.\mathsf{Self}$	$\{\mathrm{d} \mid (\mathrm{d}, \mathrm{d}) \in s^{\mathcal{I}}\}$

Table 2.1: Concepts and roles in $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$.

Any triple $\mathcal{V} \triangleq (\sigma_i, \sigma_c, \sigma_r)$ from $\mathbf{N_I} \times \mathbf{N_C} \times \mathbf{N_R}$ having finite components will be called a *vocabulary*. We say $\mathcal{L}(\mathcal{V})$-concepts for those $\mathcal{L}$-concepts that employ only symbols from $\mathcal{V}$. For a *pointed interpretation* $(\mathcal{I}, \mathrm{d})$ we say that it *satisfies* a concept C (written: $(\mathcal{I}, \mathrm{d}) \models \mathrm{C}$) if $\mathrm{d} \in \mathrm{C}^{\mathcal{I}}$. A $\mathcal{V}$-pointed-interpretation $(\mathcal{I}, \mathrm{d})$ is a partial interpretation, where all individual names outside $\mathcal{V}$ are left undefined while other symbols outside $\mathcal{V}$ are interpreted as $\emptyset$.

2.2 Preliminaries on category theory

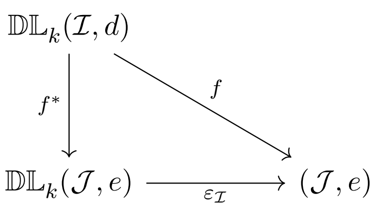
We assume familiarity with basic concepts such as categories, functors or natural transformations. For a definition of a category, functor and natural transformation, see [10, Definition 1.1, 1.2 and 7.6]. Let $\mathbb{C}$ and $\mathbb{D}$ be categories. We write $|\mathbb{C}|$ to denote morphisms (arrows) of $\mathbb{C}$ and $f \in |\mathbb{C}|$ to indicate that f is a morphism in $\mathbb{C}$.

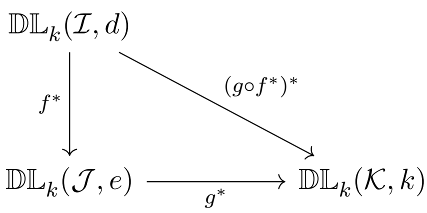
Let $G : \mathbb{C} \to \mathbb{C}$ be a functor and $\varepsilon : \mathbb{C} \Rightarrow 1_{\mathbb{C}}$ a natural transformation, with $1_{\mathbb{C}}$ being the identity functor on $\mathbb{C}$.

Definition 2.2.1. A *comonad* G is a triple $(G, \varepsilon, (\cdot)^*)$, where ε is called the *counit* of G that for each object A it gives us an arrow $\varepsilon_A : GA \to A$, while $(\cdot)^*$, called the *Kleisli coextension* of G, is an operator sending each arrow $f : GA \to B$ to $f^* : GA \to GB$.

These have to satisfy, for all $f : GA \to B$ and $g : GB \to C$, the equations:

$$\varepsilon_A^* = 1_{GA}, \qquad \varepsilon_B \circ f^* = f, \qquad (g \circ f^*)^* = g^* \circ f^*$$





Definition 2.2.2. A *coKleisli category* $\mathsf{Kl}(G)$ is a category with objects from $\mathbb{C}$ and arrows from A to B given by the arrows in $\mathbb{C}$ of the form $GA \to B$, where composition $g \bullet f$ is given by $g \circ f^*$.

We shall also need the notion of *relative comonads* [9]:

Definition 2.2.3 (Relative comonad)**.** Given a functor $J : \mathbb{C} \to \mathbb{D}$, and a comonad G on $\mathbb{D}$, we obtain a *relative comonad* on $\mathbb{C}$, whose coKleisli category is defined as follows. A morphism from A to B, for objects A, B of $\mathbb{C}$, is a $\mathbb{D}$-arrow $GJA \to JB$. The counit at A is ε_{JA}, using the counit of G at JA. Given $f : GJA \to JB$, the Kleisli coextension $f^* : GJA \to GJB$ is the Kleisli coextension of G. Since G is a comonad, these operations satisfy the equations for a comonad in Kleisli form. We write this as $(G \circ J)$-relative-comonad.

Chapter 3

Bisimulation Games

We now shall recall the characterization of the equality of interpretations under a certain logic via bisimulation games and bisimulation relation and argue their logical equivalence.

Definition 3.0.1. We write $(\mathcal{I}, \mathrm{d}) \equiv_k^{\mathcal{L}\Phi(\mathcal{V})} (\mathcal{J}, \mathrm{e})$ iff d and e satisfy the same $\mathcal{L}\Phi(\mathcal{V})$-concepts of rank at most k, where $k \in \mathbb{N} \cup \{\omega\}$.

3.1 Games

Let $\mathcal{V}$ be a vocabulary. Following [18], we recap the notion of *bisimulation games* for $\mathcal{ALC}$ and its extensions.

Definition 3.1.1. Call $\mathrm{d} \in \Delta^{\mathcal{I}}$ and $\mathrm{e} \in \Delta^{\mathcal{J}}$ to be in $\mathcal{V}$-*harmony*[1] if for all concept names $\mathrm{C} \in \sigma_c$ we have that $\mathrm{d} \in \mathrm{C}^{\mathcal{I}}$ iff $\mathrm{e} \in \mathrm{C}^{\mathcal{J}}$.

The $\mathcal{ALC}(\mathcal{V})$-*bisimulation game* is played by two players, Spoiler (he) and Duplicator (she), on two pointed interpretations $(\mathcal{I}, \mathrm{d}_0)$ and $(\mathcal{J}, \mathrm{e}_0)$. A *configuration* of a game is a quartet of the form $(\mathcal{I}, s; \mathcal{J}, s')$, where s and s' are words from, respectively, $\Delta^{\mathcal{I}}(\sigma_r\Delta^{\mathcal{I}})^*$ and $\Delta^{\mathcal{J}}(\sigma_r\Delta^{\mathcal{J}})^*$. Intuitively, configurations encode not only the current position of the play but also its full play history. The *initial configuration* is simply $(\mathcal{I}, \mathrm{d}_0; \mathcal{J}, \mathrm{e}_0)$. The 0-th round of the game starts in the initial configuration and we require that d_0 and e_0 are in $\mathcal{V}$-harmony. If not, then immediately Spoiler wins. For any configuration $(\mathcal{I}, s\mathrm{d}; \mathcal{J}, s'\mathrm{e})$ (where the sequences s, s' may be empty) in the game, the following rules apply:

(a) In each round, Spoiler picks one of the two interpretations, say $\mathcal{I}$. Then he picks a role name $r \in \sigma_r$ and takes an element $\mathrm{d}' \in \Delta^{\mathcal{I}}$ such that (♡): $(\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}$. If there is no such role name r and an element d', then Duplicator wins.

[1] For $\mathcal{ALC}$ we do not actually use σ_i and σ_r, but they will be useful for other logics.

(b) Duplicator responds in the other interpretation, $\mathcal{J}$, by picking the same role name $r \in \sigma_r$ as Spoiler did and an element $\mathrm{e}' \in \Delta^{\mathcal{I}}$ in $\mathcal{V}$-harmony with d', witnessing (♣): $(\mathrm{e}, \mathrm{e}') \in r^{\mathcal{J}}$. If there is no such role name r or an element e', Spoiler wins.

The game continues from the position $(\mathcal{I}, s\mathrm{d}r\mathrm{d}'; \mathcal{J}, s'\mathrm{e}r\mathrm{e}')$. Duplicator has a winning strategy in the game on $(\mathcal{I}, \mathrm{d}_0; \mathcal{J}, \mathrm{e}_0)$ if she can respond to every move of Spoiler so that she either wins the game or can survive ω rounds. We define winning strategies in k-round games analogously.

The above game is adjusted to the case of expressive sublogics $\mathcal{L}\Phi$ of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$ as follows.

- If $\mathcal{O} \in \Phi$, then we extend the definition of $\mathcal{V}$-harmony with a condition "for all $\mathrm{o} \in \sigma_i$ we have that $\mathrm{d} = \mathrm{o}^{\mathcal{I}}$ iff $\mathrm{e} = \mathrm{o}^{\mathcal{J}}$".
- If $\mathsf{Self} \in \Phi$, then we extend the definition of $\mathcal{V}$-harmony with a condition "for all $r \in \sigma_r$ we have that $(\mathrm{d}, \mathrm{d}) \in r^{\mathcal{I}}$ iff $(\mathrm{e}, \mathrm{e}) \in r^{\mathcal{J}}$".
- If $\mathcal{I} \in \Phi$, then in Spoiler's move the condition (♡) additionally allows for $(\mathrm{d}', \mathrm{d}) \in r^{\mathcal{I}}$. Then in the corresponding move of Duplicator, the condition (♣) imposes $(\mathrm{e}', \mathrm{e}) \in r^{\mathcal{J}}$.
- If $b \in \Phi$, then for the element e' we additionally extend (♣) to fulfil the equality $\{r \in \sigma_r \mid (\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}\} = \{r \in \sigma_r \mid (\mathrm{e}, \mathrm{e}') \in r^{\mathcal{J}}\}$. Moreover, in case of $\mathcal{I} \in \Phi$ then also $\{r \in \sigma_r \mid (\mathrm{d}', \mathrm{d}) \in r^{\mathcal{I}}\} = \{r \in \sigma_r \mid (\mathrm{e}', \mathrm{e}) \in r^{\mathcal{J}}\}$ must hold.

Proposition 3.1.2. $\mathcal{V}$-harmony *is a transitive relation under all game variations*

Proof. Notice that in the definition we have used everywhere logical equivalence, from which transitivity follows directly. Clearly combining logics together preserves that. □

The following fact for most of the considered logics is either well-known (see [18], in particular, Prop. 2.1.3 and related chapters) or can be established by tiny modifications of the existing proofs:

Fact 3.1.3. *Let $\mathcal{L}$ be a description logic satisfying $\mathcal{ALC} \subseteq \mathcal{L} \subseteq \mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$. Duplicator has a winning strategy in $\mathcal{L}(\mathcal{V})$-bisimulation game played on two pointed interpretations $(\mathcal{I}, \mathrm{d})$ and $(\mathcal{J}, \mathrm{e})$ iff $(\mathcal{I}, \mathrm{d})$ and $(\mathcal{J}, \mathrm{e})$ satisfy the same $\mathcal{L}(\mathcal{V})$-concepts.*

3.2 Bisimulations

To simplify reasoning about bisimulation games, we employ the well-known notion of *bisimulation*, which can be seen as the "encoding" of winning strategies of Duplicator. Let $\mathcal{L}\Phi$ be an expressive sublogic of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$ and $k \in \mathbb{N} \cup \{\omega\}$. Following [14]:

Definition 3.2.1 (Bisimulation relation)**.** $\mathcal{L}\Phi(\mathcal{V})$-*k-bisimulation* between $(\mathcal{I}, \mathrm{a})$ and $(\mathcal{J}, \mathrm{b})$ is a set $\mathcal{Z} \subseteq \bigcup_{\ell=0}^{k} (\Delta^{\mathcal{I}})^k \times (\Delta^{\mathcal{J}})^k$ satisfying the following seven conditions for all $\mathrm{o} \in \sigma_i, \mathrm{C} \in \sigma_c, r \in \sigma_r, \mathrm{d}, \mathrm{d}' \in \Delta^{\mathcal{I}}$, $s \in (\Delta^{\mathcal{I}})^*$ and $\mathrm{e}, \mathrm{e}' \in \Delta^{\mathcal{J}}$, $s' \in (\Delta^{\mathcal{J}})^*$:

(a) If $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ then $\mathrm{d} \in \mathrm{C}^{\mathcal{I}}$ iff $\mathrm{e} \in \mathrm{C}^{\mathcal{J}}$.

(b) If $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ and $(\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}$ then there is $\mathrm{e}' \in \Delta^{\mathcal{J}}$ s.t. $(\mathrm{e}, \mathrm{e}') \in r^{\mathcal{J}}$ and $\mathcal{Z}(s\mathrm{dd}', s'\mathrm{ee}')$.

(c) If $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ and $(\mathrm{e}, \mathrm{e}') \in r^{\mathcal{J}}$ then there is $\mathrm{d}' \in \Delta^{\mathcal{I}}$ s.t. $(\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}$ and $\mathcal{Z}(s\mathrm{dd}', s'\mathrm{ee}')$.

(d) If $\mathcal{O} \in \Phi$, then $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ implies $\mathrm{d} = \mathrm{o}^{\mathcal{I}}$ iff $\mathrm{e} = \mathrm{o}^{\mathcal{J}}$.

(e) If $\mathsf{Self} \in \Phi$, then $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ implies $(\mathrm{d}, \mathrm{d}) \in r^{\mathcal{I}}$ iff $(\mathrm{e}, \mathrm{e}) \in r^{\mathcal{J}}$.

(f) If $\mathcal{I} \in \Phi$, then $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ and $(\mathrm{d}', \mathrm{d}) \in r^{\mathcal{I}}$ implies that there is $\mathrm{e}' \in \Delta^{\mathcal{J}}$ such that $(\mathrm{e}', \mathrm{e}) \in r^{\mathcal{J}}$ and $\mathcal{Z}(s\mathrm{dd}', s'\mathrm{ee}')$.

(g) If $b \in \Phi$,, then if $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ and $(\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}$ implies that there is $\mathrm{e}' \in \Delta^{\mathcal{J}}$ satisfying $\mathcal{Z}(s\mathrm{dd}', s'\mathrm{ee}')$ and $\{r \in \sigma_r \mid (\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}\} = \{r \in \sigma_r \mid (\mathrm{e}, \mathrm{e}') \in r^{\mathcal{J}}\}$. If $\mathcal{I} \in \Phi$, then also $\{r \in \sigma_r \mid (\mathrm{d}', \mathrm{d}) \in r^{\mathcal{I}}\} = \{r \in \sigma_r \mid (\mathrm{e}', \mathrm{e}) \in r^{\mathcal{J}}\}$.

Note that if $\mathcal{Z}$ is an ω-bisimulation, then $\mathcal{Z}$ becomes a k-bisimulation when restricted to pairs of sequences of length at most k. 2.1.3 from [18]) that: The following fact for most of the considered logics is either well-known (see [18], in particular, Prop. 2.1.3 and related chapters) or can be established by tiny modifications of existing proofs.

Fact 3.2.2. *For any $k \in \mathbb{N} \cup \{\omega\}$ and a logic $\mathcal{L}\Phi$ between $\mathcal{ALC}$ and $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$, t.f.a.e.:*

- *Duplicator has the winning strategy in the k-round $\mathcal{L}\Phi(\mathcal{V})$-bisimulation-game on $(\mathcal{I}, \mathrm{d}; \mathcal{J}, \mathrm{e})$,*
- *There is an $\mathcal{L}\Phi(\mathcal{V})$-$k$-bisimulation $\mathcal{Z}$ between $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$ such that $\mathcal{Z}(\mathrm{d}, \mathrm{e})$,*
- $(\mathcal{I}, \mathrm{d}) \equiv_k^{\mathcal{L}\Phi(\mathcal{V})} (\mathcal{J}, \mathrm{e})$.

Chapter 4

Reductions between games and logics

Herein we establish reductions, based on appropriate model transformations, that will allow us to transfer the winning strategies of Duplicator from richer logics to weaker ones, ending up on $\mathcal{ALC}$. All of them, except the case of nominals, will be trivial. Such transformation will be essential in Chapter 6, where we shall employ them in the construction of relative comonads.

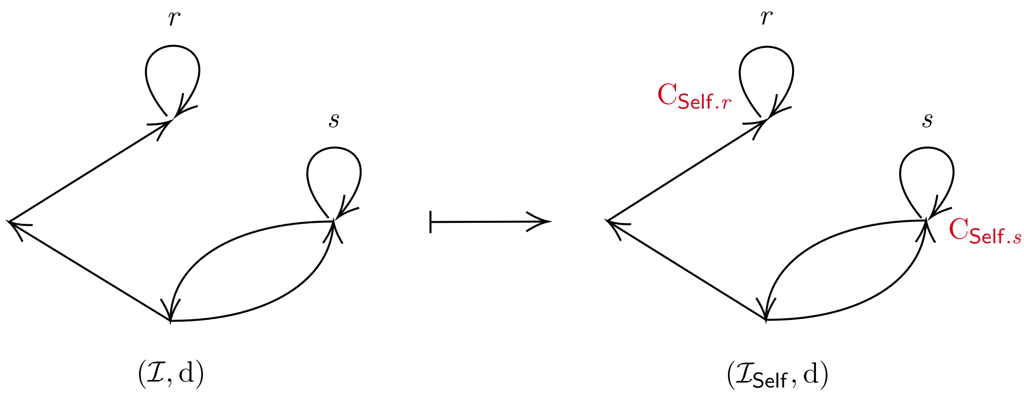
We will denote the game reductions for logic extensions $\Phi \subseteq \{\mathsf{Self}, \mathcal{I}, b, \mathcal{O}\}$ by $\mathfrak{f}_\Phi$, which has two components $\mathfrak{f}^{\mathcal{I}}_\Phi$ and $\mathfrak{f}^{*}_\Phi$, that define actions on, respectively, the interpretation and the distinguished element.

4.1 A family of logic reductions

Definition 4.1.1. Let $\mathcal{I}$ be an interpretation over vocabulary $(\sigma_i, \sigma_c, \sigma_r)$. A $(\sigma'_i, \sigma'_c, \sigma'_r)$-*reduct* of an interpretation $\mathcal{I}$ is an interpretation $\mathcal{I}'$ obtained by interpreting all the symbols outside of $\sigma'_i \cup \sigma'_c \cup \sigma'_r$ as empty sets.

4.1.1 Self operator

We first handle the Self operator. Let $\sigma_c^{\mathsf{Self}} \triangleq \sigma_c \cup \{\mathrm{C}_{\mathsf{Self}.r} \mid r \in \sigma_r\}$. By the *self-enrichment* of a $\mathcal{V} \triangleq (\sigma_i, \sigma_c, \sigma_r)$-interpretation $\mathcal{I}$ we mean the $\mathcal{V}^{\mathsf{Self}} \triangleq (\sigma_i, \sigma_c^{\mathsf{Self}}, \sigma_r)$-interpretation $\mathcal{I}_{\mathsf{Self}}$, where the $(\sigma_i, \sigma_c, \sigma_r)$-reduct of $\mathcal{I}_{\mathsf{Self}}$ is equal to $\mathcal{I}$ and the interpretations of $\mathrm{C}_{\mathsf{Self}.r}$ concepts are defined as $(\mathrm{C}_{\mathsf{Self}.r})^{\mathcal{I}_{\mathsf{Self}}} = (\exists r.\mathsf{Self})^{\mathcal{I}}$.



Let $\mathfrak{f}_{\mathsf{Self}}$ be the described transformation, mapping $(\mathcal{I}, \mathrm{d})$ to $(\mathcal{I}_{\mathsf{Self}}, \mathrm{d})$.

Proposition 4.1.2. *Let $k \in \mathbb{N} \cup \{\omega\}$ and let $\mathcal{L}$ be a DL satisfying $\mathcal{ALC} \subseteq \mathcal{L} \subseteq \mathcal{ALCIbO}$. Then Duplicator has a winning strategy in a k-round $\mathcal{L}_{\mathsf{Self}}(\mathcal{V})$-bisimulation game on $(\mathcal{I}, \mathrm{d}; \mathcal{J}, \mathrm{e})$ iff she has a winning strategy in a k-round $\mathcal{L}(\mathcal{V})$-bisimulation game on $(\mathfrak{f}_{\mathsf{Self}}(\mathcal{I}), \mathrm{d}; \mathfrak{f}_{\mathsf{Self}}(\mathcal{J}), \mathrm{e})$.*

Proof. By applying Fact 3.2.2 to both sides, it is sufficient to prove the following:

There is a $\mathcal{L}_{\mathsf{Self}}(\mathcal{V})$-$k$-bisimulation $\mathcal{Z}$ between $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$ such that $\mathcal{Z}(d, e)$ iff there is a $\mathcal{L}(\mathcal{V}^{\mathsf{Self}})$-$k$-bisimulation $\mathcal{Z}_{\mathsf{Self}}$ between $(\mathfrak{f}_{\mathsf{Self}}(\mathcal{I}), \mathrm{d})$ and $(\mathfrak{f}_{\mathsf{Self}}(\mathcal{J}), \mathrm{e})$ such that $\mathcal{Z}_{\mathsf{Self}}(d, e)$

$(\Longrightarrow)$ Let us assume $\mathcal{Z}$ is the bisimulation from implication predecessor and take $\mathcal{Z}_{\mathsf{Self}} \triangleq \mathcal{Z}$. We now need to prove that $\mathcal{Z}_{\mathsf{Self}}$ is a valid bisimulation. Notice that the only way in which $\mathfrak{f}_{\mathsf{Self}}$-reduced interpretations differ are the atomic concepts, so it is sufficient to prove that case (a) from Definition 3.2.1 holds for new $\mathrm{C}_{\mathsf{Self}.r}$ concepts. Take any $a \in \mathcal{I}, b \in \mathcal{J}$.

$$
\begin{aligned}
\mathcal{Z}_{\mathsf{Self}}(a,b) &\Longrightarrow \mathcal{Z}(a,b) && \mathcal{Z}_{\mathsf{Self}} = \mathcal{Z} \\
&\Longrightarrow (a,a) \in r^{\mathcal{I}} \iff (b,b) \in r^{\mathcal{J}} && \text{def. } \mathcal{Z},\ (e) \\
&\Longrightarrow a \in (\exists r.\mathsf{Self})^{\mathcal{I}} \iff b \in (\exists r.\mathsf{Self})^{\mathcal{J}} && \text{def. } \exists r.\mathsf{Self} \\
&\Longrightarrow a \in (\mathrm{C}_{\mathsf{Self}.r})^{\mathcal{I}_{\mathsf{Self}}} \iff b \in (\mathrm{C}_{\mathsf{Self}.r})^{\mathcal{J}_{\mathsf{Self}}} && \text{def. } \mathrm{C}_{\mathsf{Self}.r}
\end{aligned}
$$

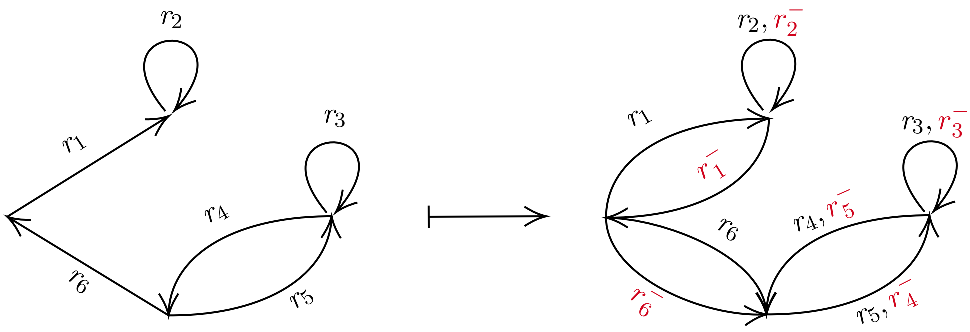
$(\Longleftarrow)$ Proof for the other side is analogous. Let us again assume $\mathcal{Z}_{\mathsf{Self}}$ is the bisimulation from implication predecessor and take $\mathcal{Z} \triangleq \mathcal{Z}_{\mathsf{Self}}$. We now need to prove that $\mathcal{Z}$ is a valid bisimulation. This time, the only case that needs special attention is (e) from Definition 3.2.1. Take any $a \in \mathcal{I}, b \in \mathcal{J}$.

$$\begin{aligned}
\mathcal{Z}(a,b) &\Longrightarrow \mathcal{Z}_{\mathsf{Self}}(a,b) && \mathcal{Z} = \mathcal{Z}_{\mathsf{Self}} \\
&\Longrightarrow a \in (\mathrm{C}_{\mathsf{Self}.r})^{\mathcal{I}_{\mathsf{Self}}} \iff b \in (\mathrm{C}_{\mathsf{Self}.r})^{\mathcal{J}_{\mathsf{Self}}} && \text{def. } \mathcal{Z},\ (a) \\
&\Longrightarrow a \in (\exists r.\mathsf{Self})^{\mathcal{I}} \iff b \in (\exists r.\mathsf{Self})^{\mathcal{J}} && \text{def. } \mathrm{C}_{\mathsf{Self}.r} \\
&\Longrightarrow (a,a) \in r^{\mathcal{I}} \iff (b,b) \in r^{\mathcal{J}} && \text{def. } \exists r.\mathsf{Self}
\end{aligned}$$

□

4.1.2 Role inverses

Our next goal is to incorporate inverses of roles. Let $\sigma_r^{\mathcal{I}} \triangleq \sigma_r \cup \{r_{inv} \mid r \in \sigma_r\}$ By the *inverse-enrichment* of a $\mathcal{V} \triangleq (\sigma_i, \sigma_c, \sigma_r)$-interpretation $\mathcal{I}$ we mean the $\mathcal{V}^{\mathcal{I}} \triangleq (\sigma_i, \sigma_c, \sigma_r^{\mathcal{I}})$-interpretation $\mathcal{I}_{\mathcal{I}}$, where the $(\sigma_i, \sigma_c, \emptyset)$-reducts of $\mathcal{I}$ and $\mathcal{I}_{\mathcal{I}}$ are equal, and the interpretations of role names r_{inv} are defined as $(r_{inv})^{\mathcal{I}_{\mathcal{I}}} = (r^{-})^{\mathcal{I}}$.



$(\mathcal{I}, \mathrm{d})$ $(\mathcal{I}_{\mathcal{I}}, \mathrm{d})$

Let $\mathfrak{f}_{\mathcal{I}}$ be the described transformation, mapping $(\mathcal{I}, \mathrm{d})$ to $(\mathcal{I}_{\mathcal{I}}, \mathrm{d})$. The proposition follows in a similar pattern to Proposition 4.1.2:

Proposition 4.1.3. *Let $k \in \mathbb{N} \cup \{\omega\}$ and let $\mathcal{L}$ be a DL satisfying $\mathcal{ALC} \subseteq \mathcal{L} \subseteq \mathcal{ALCO}b$. Then Duplicator has a winning strategy in a k-round $\mathcal{LI}(\mathcal{V})$-bisimulation game on $(\mathcal{I}, \mathrm{d}; \mathcal{J}, \mathrm{e})$ iff she has a winning strategy in a k-round $\mathcal{L}(\mathcal{V}^{\mathcal{I}})$-bisimulation game on $(\mathfrak{f}_{\mathcal{I}}(\mathcal{I}), \mathrm{d}; \mathfrak{f}_{\mathcal{I}}(\mathcal{J}), \mathrm{e})$.*

Proof. By applying Fact 3.2.2 to both sides, it is sufficient to prove the following:

There is a $\mathcal{L}_{\mathcal{I}}(\mathcal{V})$-$k$-bisimulation $\mathcal{Z}$ between $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$ such that $\mathcal{Z}(d, e)$ iff there is a $\mathcal{L}(\mathcal{V}^{\mathcal{I}})$-$k$-bisimulation $\mathcal{Z}_{\mathcal{I}}$ between $(\mathfrak{f}_{\mathcal{I}}(\mathcal{I}), \mathrm{d})$ and $(\mathfrak{f}_{\mathcal{I}}(\mathcal{J}), \mathrm{e})$ such that $\mathcal{Z}_{\mathcal{I}}(d, e)$

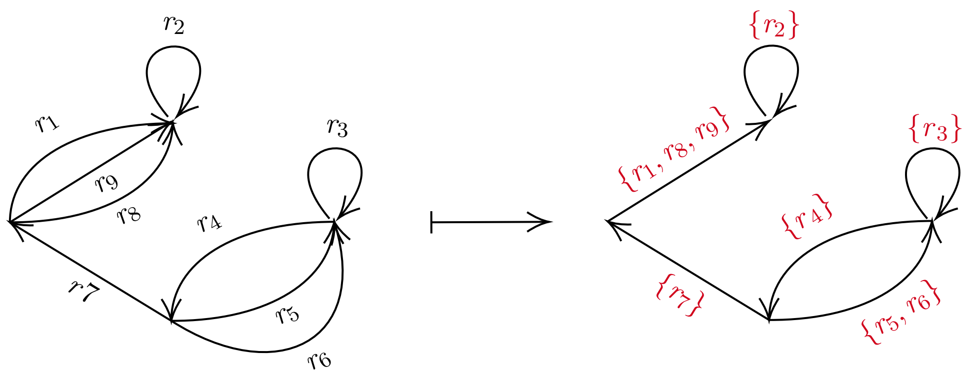
($\Longrightarrow$) Let us assume $\mathcal{Z}$ is the bisimulation from implication predecessor and take $\mathcal{Z}_{\mathcal{I}} \triangleq \mathcal{Z}$. Notice that the only way in which $\mathfrak{f}_{\mathcal{I}}$-reduced interpretations differ are the added fresh inverse roles, so it is sufficient to prove that cases (b) and (c) from Definition 3.2.1 hold for $\sigma_r^{\mathcal{I}}$ roles. The case for roles in σ_r is trivial, as there were no changes to them made and we have that $\mathcal{Z}_{\mathcal{I}} = \mathcal{Z}$. Take $a, a' \in \mathcal{I}, b \in \mathcal{J}, r^- \in \sigma_r^{\mathcal{I}} \setminus \sigma_r$ and assume that $\mathcal{Z}_{\mathcal{I}}(a, b)$ and $(a', a) \in r^{-\mathcal{I}}$. Let us consider the case (b), case (c) will follow analogously. We need to show that there exists $b' \in \mathcal{J}$ s.t. $(b', b) \in r^{-\mathcal{J}}$ and $\mathcal{Z}_{\mathcal{I}}(aa', bb')$. By construction, r^- has a corresponding role r s.t. $(a, a') \in r^{\mathcal{I}}$. From $\mathcal{Z}(a, b)$ assumption, we can extract b' s.t. $(b, b') \in r^{\mathcal{J}}$. By definition of the construction, this implies that $(b', b) \in r^{-\mathcal{J}}$ which closes the proof.

$(\Longleftarrow)$ Proceeds similarly as the proof above.

□

4.1.3 Safe boolean roles combinations

We focus next on safe boolean combinations of roles. Given a finite $\sigma_r \subseteq \mathbf{N_R}$, let σ_r^b be composed of role names having the form r_S, where S is any non-empty subset of σ_r. By the *b-enrichment* of a $\mathcal{V} \triangleq (\sigma_i, \sigma_c, \sigma_r)$-interpretation $\mathcal{I}$ we mean the $\mathcal{V}^b \triangleq (\sigma_i, \sigma_c, \sigma_r^b)$-interpretation $\mathcal{I}_b$, where the $(\sigma_i, \sigma_c, \emptyset)$-reducts of $\mathcal{I}$ and $\mathcal{I}_b$ are equal and the interpretation of role names $r_S \in \sigma_r^b$ is defined as $\{(\mathrm{d}, \mathrm{e}) \mid S = \{r \in \sigma_r \mid (\mathrm{d}, \mathrm{e}) \in r^{\mathcal{I}}\}\}$.



$(\mathcal{I}, \mathrm{d})$ $(\mathcal{I}_b, \mathrm{d})$

Let $\mathfrak{f}_b$ be the described transformation, mapping $(\mathcal{I}, \mathrm{d})$ to $(\mathcal{I}_b, \mathrm{d})$. Once more, the following proposition is straightforward:

Proposition 4.1.4. *Let $k \in \mathbb{N} \cup \{\omega\}$ and let $\mathcal{L}$ be a DL satisfying $\mathcal{ALC} \subseteq \mathcal{L} \subseteq \mathcal{ALCO}$. Then Duplicator has a winning strategy in a k-round $\mathcal{L}b(\mathcal{V})$-bisimulation-game on $(\mathcal{I}, \mathrm{d}; \mathcal{J}, \mathrm{e})$ iff she has a winning strategy in a k-round $\mathcal{L}(\mathcal{V}^b)$-bisimulation-game on $(\mathfrak{f}_b(\mathcal{I}), \mathrm{d}; \mathfrak{f}_b(\mathcal{J}), \mathrm{e})$.*

Proof. The key observation here is that safe boolean roles combinations are giving us the power to define any 2-type as a step in the bisimulation. Henceforth, we convert the interpretation such that the arrows represent exactly 2-types and therefore a move in the game can cover any move that could have been expressed by roles combinations. A detailed proof is very similar to Proposition 4.1.2 and Proposition 4.1.3 and thus shall be left as an exercise for the reader. □

4.1.4 Nominals

Finally, we proceed with the case of nominals. In this case, we need to be extra careful, as the comonads introduced in the next section will act as unravelling on interpretations, and we do not want to create multiple copies of a nominal. Recall that the Gaifman graph $\mathsf{G}_{\mathcal{I}} = (V_{\mathcal{I}}, E_{\mathcal{I}})$ of an interpretation $\mathcal{I}$ is a simple undirected graph whose nodes are domain elements from $\Delta^{\mathcal{I}}$ and an edge exists between two nodes when there is a role that connects them in $\mathcal{I}$.

Let $\sigma_c^{\mathcal{O}} \triangleq \sigma_c \cup \{\mathrm{C}_{\mathrm{o},r} \mid \mathrm{o} \in \sigma_i, r \in \sigma_r\}$ and $\sigma_r^{\mathcal{O}} \triangleq \sigma_r \cup \{r_{\mathrm{o}} \mid \mathrm{o} \in \sigma_i\}$. By the *nominal-enrichment* of a $\mathcal{V} \triangleq (\sigma_i, \sigma_c, \sigma_r)$-interpretation $\mathcal{I}$ we mean the $\mathcal{V}^{\mathcal{O}} \triangleq (\sigma_i, \sigma_c^{\mathcal{O}}, \sigma_r^{\mathcal{O}})$-interpretation $\mathcal{I}_{\mathcal{O}}$ defined in the following steps. We encourage the reader to consult the example depicted below while going through the steps:

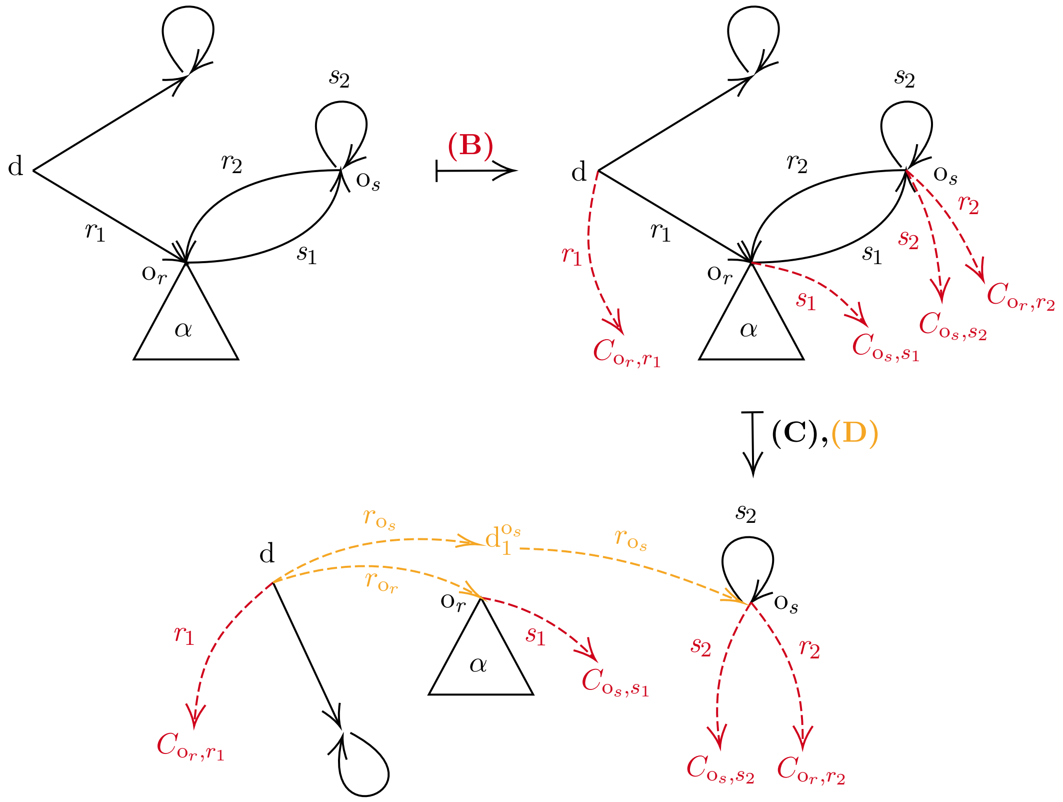
- **(A)** First, we get rid of unreachable elements from $\mathcal{I}$. More precisely, let $\mathcal{J}$ to be the substructure of $\mathcal{I}$ restricted to the set of all elements reachable in (finitely-many steps) from d in $\mathsf{G}_{\mathcal{I}}$. Without the loss of generality, we can assume that all $\mathrm{o}^{\mathcal{I}}$ for $\mathrm{o} \in \sigma_i$ are reachable.

- **(B)** For each pair $(\mathrm{d}, \mathrm{o}) \in \Delta^{\mathcal{I}} \times \sigma_i$ such that there is a r-connection from d to $\mathrm{o}^{\mathcal{I}}$, we insert a "trampoline" element labelled by the unique concept name $\mathrm{C}_{\mathrm{o},r}$ and we r-connect it with d.

  Trampoline elements are used to bookkeep information about connections between elements and named elements. Let $\mathcal{J}$ be the resulting interpretation.

- **(C)** We next divide $\mathcal{J}$ into components. Let $\mathcal{J}_{\mathrm{o}}$ for $\mathrm{o} \in \sigma_i \cup \{\mathrm{d}\}$ (with d being the root element) be induced subinterpretations of $\mathcal{J}$ obtained by removing all elements $\{\mathrm{o}^{\mathcal{I}} \mid \mathrm{o} \in \sigma_i\}$ from $\mathcal{J}$ except the element mentioned in the subscript (that serve the role of distinguished elements of the components). In each component $\mathcal{J}_{\mathrm{o}}$, we take only elements reachable from o. Take $\mathcal{J}'$ to be the disjoint sum of the components.

- **(D)** In the last step, we will link components. For all $\mathrm{o} \in \sigma_i$, take $\mathrm{dist}_{\mathrm{o}}$ to be the length of the shortest path from d to $\mathrm{o}^{\mathcal{I}}$ in $\mathsf{G}_{\mathcal{I}}$. We will connect d to $\mathrm{o}^{\mathcal{J}'}$ by a dummy path of length precisely $\mathrm{dist}_{\mathrm{o}}$. Thus, we introduce dummy elements $\mathrm{d}_1^{\mathrm{o}}, \ldots, \mathrm{d}_{\mathrm{dist}_{\mathrm{o}}-1}^{\mathrm{o}}$ to $\Delta^{\mathcal{J}'}$ and employ the fresh role name r_{o}, whose interpretation will contain precisely the pairs $(\mathrm{d}, \mathrm{d}_1^{\mathrm{o}}), (\mathrm{d}_1^{\mathrm{o}}, \mathrm{d}_2^{\mathrm{o}}), \ldots, (\mathrm{d}_{\mathrm{dist}_{\mathrm{o}}-1}^{\mathrm{o}}, \mathrm{o}^{\mathcal{J}'})$. The resulting interpretation is the desired $\mathcal{I}_{\mathcal{O}}$.



Let $\mathfrak{f}_{\mathcal{O}}$ be the described transformation, mapping $(\mathcal{I}, \mathrm{d})$ to $(\mathcal{I}_{\mathcal{O}}, \mathrm{d})$.

Lemma 4.1.5. *Let $k \in \mathbb{N} \cup \{\omega\}$. Duplicator has a winning strategy in a k-round $\mathcal{ALCO}(\mathcal{V})$-bisimulation game on $(\mathcal{I}, \mathrm{d})$ and $(\mathcal{J}, \mathrm{e})$ iff she has a winning strategy in a k-round $\mathcal{ALC}(\mathcal{V}^{\mathcal{O}})$-bisimulation game on $(\mathfrak{f}_{\mathcal{O}}(\mathcal{I}), \mathrm{d})$ and $(\mathfrak{f}_{\mathcal{O}}(\mathcal{J}), \mathrm{e})$.*

***Proof** ($\Longrightarrow$).* We proceed with the proof by induction on k, the depth parameter. Interpretation of concept names for distinguished elements is left unchanged by $\mathfrak{f}_{\mathcal{O}}$, hence Duplicator has a winning strategy in the 0-round bisimulation game. Suppose now that the implication holds for games with at most k rounds and let us show it holds for games with k+1 rounds. Suppose that Duplicator has a winning strategy in any k+1-round $\mathcal{ALCO}(\mathcal{V})$-bisimulation game. Let $(\mathfrak{f}_{\mathcal{O}}(\mathcal{I}), s\mathrm{d}; \mathfrak{f}_{\mathcal{O}}(\mathcal{J}), s'\mathrm{e})$ be a configuration of the $\mathcal{ALC}(\mathcal{V}^{\mathcal{O}})$-bisimulation game following the promised (by inductive hypothesis) k-round winning strategy of Duplicator. We will show how to proceed with the next step of the game. W.l.o.g. assume that Spoiler selected $\mathfrak{f}_{\mathcal{O}}(\mathcal{I})$ and decided to choose an element d'; we need to reply with an element e' in the second structure. There are the following cases:

1. Spoiler chooses a dummy element. We reply with the corresponding element, which can be done without any problems since dummy paths of length at most

$k{+}1$ leading to named elements have equal lengths in both interpretations. Dummy paths longer than $k{+}1$ are clearly equal up to $k{+}1$ elements.

2. d' selected by Spoiler is a trampoline. Notice that we have defined the trampolines in such a way that they reflect all possible connections to constants. Hence, by having $k{+}1$ rounds winning strategy in $\mathcal{ALCO}(\mathcal{V})$-bisimulation game, it implies that the elements reachable within k steps must have had the same connections to constants, which means that Duplicator can respond with a trampoline of equal concept names.

3. Spoiler chooses a constant $\mathrm{o}^{\mathcal{I}}$. The only way which we could access a constant was via a dummy path of length at most k, which means that d, e were on the paths labelled by the r_{o}, thus they lead to the same constants, $\mathrm{o}^{\mathcal{I}}$ and $\mathrm{o}^{\mathcal{J}}$, respectively.

4. Spoiler chooses an "ordinary" element d', that is, an element which does not match any of the above conditions. Then it means that d' was a copy of an element in the original interpretation, thus, we can follow the same move that was made in the original interpretation by $\mathcal{ALCO}(\mathcal{V})$-winning strategy.

□

($\Longleftarrow$). We again proceed by induction on k. The base case proceeds analogously to the previous implication. Suppose now that the implication holds for games with at most k rounds and let us show it holds for games with $k{+}1$ rounds. Suppose that Duplicator has a winning strategy in any $k{+}1$-round $\mathcal{ALC}(\mathcal{V}^{\mathcal{O}})$-bisimulation game. Let $(\mathcal{I}s\mathrm{d};\mathcal{J}s'\mathrm{e})$ be a configuration of the $\mathcal{ALCO}(\mathcal{V})$-bisimulation game following the promised (by inductive hypothesis) k-round winning strategy of Duplicator. We will show how to proceed with the next step of the game. W.l.o.g. assume that Spoiler selected $\mathcal{I}$ and decided to choose an element d'; we need to reply with an element e' in the second structure. There are the following cases:

1. Spoiler chooses a constant $\mathrm{o}^{\mathcal{I}}$ via role r. From $\mathcal{ALC}(\mathcal{V}^{\mathcal{O}})$-winning strategy, this means that in the $\mathfrak{f}_{\mathcal{O}}(\mathcal{I})$ there must have been a trampoline which encodes the possible connections to a constant, thus there was also a trampoline in $\mathfrak{f}_{\mathcal{O}}(\mathcal{J})$ with the same concept names, which implies that there are the same connections to constants from d and e, hence, Duplicator can choose a constant $\mathrm{o}^{\mathcal{J}}$ using also r.

2. Spoiler jumps out of the constant, *i.e.* he was in $\mathrm{o}^{\mathcal{I}}$ and now using role r selects d' that is not a constant. Should $\mathrm{o}^{\mathcal{I}}$ be accessible within k steps, it means that we can access it in $\mathfrak{f}_{\mathcal{O}}(\mathcal{I})$ using a dummy path of length $\leq k$. The outgoing connections from constants were restored in $\mathfrak{f}_{\mathcal{O}}(\mathcal{I})$, henceforth, from the constant $\mathrm{o}^{\mathfrak{f}_{\mathcal{O}}(\mathcal{I})}$ we also have a r connection to a copy of the element d'. This implies that according to $\mathcal{ALC}(\mathcal{V}^{\mathcal{O}})$-winning strategy, we have a r move

to an element e' in $\mathfrak{f}_{\mathcal{O}}(\mathcal{I})$. Since e' cannot be a constant, it is a direct copy of an element from $\mathcal{J}$, which gives us a valid response for Duplicator.

3. Spoiler chooses an "ordinary" element d', that is, an element which does not match any of the above conditions. Notice that this means neither d nor d' can be a constant. That means that we have a copy of both of the elements d and d' along with all the connections between them, which means that Duplicator can respond following the $k{+}1$ step of the $\mathcal{ALC}(\mathcal{V}^{\mathcal{O}})$-winning strategy.

□

4.2 Combining reductions

We wrap up the above reductions, with a goal that the winning strategy of Duplicator in a $\mathcal{L}\Phi$-bisimulation game is equivalent to the winning strategy in a certain $\mathcal{ALC}$-bisimulation game. Note that the order of applications of reduction matters, *e.g.* we should apply first the $\mathfrak{f}_{\mathcal{I}}$ reduction, and only then $\mathfrak{f}_b$; otherwise we will not get all possible combinations of roles with inverses. Hence, we first proceed with $\mathfrak{f}_{\mathsf{Self}}$ reduction, then with $\mathfrak{f}_{\mathcal{I}}$, with $\mathfrak{f}_b$ and finally with $\mathfrak{f}_{\mathcal{O}}$. Let $\mathfrak{f}_{\Phi}$ be a composition of reductions for extensions $\Phi \in \{\mathsf{Self}, \mathcal{I}, b, \mathcal{O}\}$ in the above order.

Theorem 4.2.1. *Let $k \in \mathbb{N} \cup \{\omega\}$ and $\mathcal{L}\Phi$ satisfy $\mathcal{ALC} \subseteq \mathcal{L}\Phi \subseteq \mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$. Then Duplicator has a winning strategy in a k-round $\mathcal{L}\Phi(\mathcal{V})$-bisimulation game on $(\mathcal{I}, \mathrm{d})$ and $(\mathcal{J}, \mathrm{e})$ iff she has a winning strategy in a k-round $\mathcal{L}(\mathcal{V}^{\Phi})$-bisimulation game on $(\mathfrak{f}_{\Phi}(\mathcal{I}), \mathrm{d})$ and $(\mathfrak{f}_{\Phi}(\mathcal{J}), \mathrm{e})$.*

Proof. The key idea here is grounded on the composition of the reduction functions. Given Φ, we simply apply consecutively Propositions 4.1.2–4.1.4 and Lemma 4.1.5. □

Chapter 5

Game Comonads

Having defined a family of game reductions, we are going to start employing basic category theory primitives to define denotational semantics for bisimulation games. In this chapter, we focus on vanilla $\mathcal{ALC}$. Since $\mathcal{ALC}$ is a notational variant of the multi-modal logic, it suffices to translate the work done in [8] to the description logic setting. Subsequently, we prove that such a definition of a generalised game coincides with our definition of $\mathcal{ALC}(\mathcal{V})$-bisimulation game defined in Chapter 3. This chapter may be a bit heavy for readers not familiar enough with category theory.

The setting. In what follows, we shall work in the category of pointed interpretations $\mathcal{R}_*(\mathcal{V})$ over a vocabulary $\mathcal{V}$, where objects $(\mathcal{I}, d)$ are $\mathcal{V}$-pointed-interpretations, and morphisms $h : (\mathcal{I}, d) \to (\mathcal{J}, e)$ are homomorphisms between interpretations that preserve the distinguished element, *i.e.* $h\,d = e$. With $\mathbb{DL}_k^{\Phi}$, we will denote the corresponding game comonad, where k is the depth parameter and $\Phi \subseteq \{\mathsf{Self}, \mathcal{I}, b, \mathcal{O}\}$ parametrizes the set of language extensions. We will be a bit careless and write $\mathbb{DL}_k^{\mathcal{IO}}$ in place of $\mathbb{DL}_k^{\{\mathcal{I},\mathcal{O}\}}$, or likewise, $\mathbb{DL}_k$ to denote $\mathbb{DL}_k^{\{\}}$.

5.1 A comonad for $\mathcal{ALC}$

We start with introducing the comonad for $\mathcal{ALC}$, which will be the base for the further ones.

Definition 5.1.1 ($\mathcal{ALC}$-comonad)**.** For every $k \geq 0$, we define a comonad $\mathbb{DL}_k$ on $\mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$,[1] where $\mathbb{DL}_k$ unravels[2] $(\mathcal{I}, \mathrm{d})$ from d, up to depth k. More precisely:

- The domain of $\mathbb{DL}_k(\mathcal{I}, d)$ is composed of sequences $[a_0, r_0, a_1, r_2, \ldots] \in \Delta^{\mathcal{I}}(\sigma_r \Delta^{\mathcal{I}})^*$, where we additionally require that $(a_i, a_{i+1}) \in r_i^{\mathcal{I}}$ and $a_0 = \mathrm{d}$. The singleton sequence [d] serves as the distinguished element of $\mathbb{DL}_k(\mathcal{I}, d)$.

[1] Notice $\emptyset$ in place of σ_i. This is because $\mathcal{ALC}$-concepts cannot speak about individual names.
[2] For the notion of unravelling consult e.g. [11, Definition 3.21].

- The functorial action on morphisms for $\mathbb{DL}_k$ satisfies:

$$\mathbb{DL}_k(h : (\mathcal{I}, d) \to (\mathcal{J}, e)) : \mathbb{DL}_k(\mathcal{I}, d) \to \mathbb{DL}_k(\mathcal{J}, e)$$
$$(\mathbb{DL}_k\, h)[a_0, \alpha_1, a_1, ..., \alpha_j, a_j] = [h\, a_0, \alpha_1, h\, a_1, ..., \alpha_j, h\, a_j]$$

- The map $\varepsilon_{\mathcal{I}} : \mathbb{DL}_k(\mathcal{I}, d) \to (\mathcal{I}, d)$ sends a sequence to its last element.
- Concept names $\mathrm{C} \in \sigma_c$ are interpreted such that $s \in \mathrm{C}^{\mathbb{DL}_k(\mathcal{I},d)}$ iff $\varepsilon_{\mathcal{I}} s \in \mathrm{C}^{\mathcal{I}}$.
- For role names $r \in \sigma_r$, we put $(s, t) \in r^{\mathbb{DL}_k(\mathcal{I},d)}$ iff there is $d' \in \Delta^{\mathcal{I}}$ so that $t = s[r, d']$.
- For a morphism $h : \mathbb{DL}_k(\mathcal{I}, d) \to (\mathcal{J}, e)$, we define Kleisli coextension $h^* : \mathbb{DL}_k(\mathcal{I}, d) \to \mathbb{DL}_k(\mathcal{J}, e)$ recursively by $h^*[d] = [e]$ and $h^*(s[\alpha, d']) = h^*(s)[\alpha, h(s[\alpha, d'])])$.

Having defined the structure, we now need to prove that it indeed forms a comonad in the category-theoretic sense. We shall prove that $\mathbb{DL}_k$ is a functor, ε and $(\cdot)^*$ behave well and that the triple $(\mathbb{DL}_k, \varepsilon, (\cdot)^*)$ fulfils the comonad laws. We start with a small lemma that shall be used later in the proofs:

Lemma 5.1.2. *The following diagram in $\mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$ category commutes*

$$\begin{array}{ccc}
\mathbb{DL}_k(\mathcal{I}, d) & \xrightarrow{\mathbb{DL}_k h} & \mathbb{DL}_k(\mathcal{J}, e) \\
\downarrow{\scriptstyle \varepsilon_{\mathcal{I}}} & & \downarrow{\scriptstyle \varepsilon_{\mathcal{J}}} \\
A & \xrightarrow[h]{} & B
\end{array}$$

Proof. Let $s = [a_0, \alpha_1, a_1, ..., \alpha_j, a_j] \in \mathbb{DL}_k(\mathcal{I}, d)$. Then

$$\begin{aligned}
h(\varepsilon_{\mathcal{I}}\, s) &= h\, a_j && \text{def. } \varepsilon_{\mathcal{I}} \\
&= \varepsilon_{\mathcal{J}}[h\, a_j] && \text{def. } \varepsilon_{\mathcal{J}} \\
&= \varepsilon_{\mathcal{J}}[h\, a_0, \alpha_1, h\, a_1, ..., \alpha_j, h\, a_j] && \text{def. } \varepsilon_{\mathcal{J}} \\
&= \varepsilon_{\mathcal{J}}(\mathbb{DL}_k\, h\, s) && \text{def. } \mathbb{DL}_k\, h
\end{aligned}$$

□

Proposition 5.1.3. $\mathbb{DL}_k$ *is a functor*

Proof. We need to prove two properties

(1) $\mathbb{DL}_k$ maps objects to objects and morphisms to morphisms.

Objects. For an interpretation $\mathcal{I}$, its unravelling $\mathbb{DL}_k(\mathcal{I}, d)$ is also an interpretation over $(\sigma_i, \sigma_c, \sigma_r)$ which follows from the standard results (see e.g. [11, Definition 3.21]).

Morphisms. Suppose $h : \mathcal{I} \to \mathcal{J} \in |\mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)|$ and $s, t \in \mathbb{DL}_k(\mathcal{I}, d)$.

$$
\begin{aligned}
(s,t) \in r_\alpha^{\mathbb{DL}_k(\mathcal{I},d)} &\iff (\varepsilon_\mathcal{I}\, s, \varepsilon_\mathcal{I} t) \in r_\alpha^\mathcal{I} && \text{def. } r_\alpha^{\mathbb{DL}_k(\mathcal{I},d)} \\
&\implies (h(\varepsilon_\mathcal{I}\, s), h\, (\varepsilon_\mathcal{I} t)) \in r_\alpha^\mathcal{J} && \text{h is homomorphism} \\
&\iff (\varepsilon_\mathcal{I}(\mathbb{DL}_k\, h\, s), \varepsilon_\mathcal{I}(\mathbb{DL}_k\, h\, t)) \in r_\alpha^\mathcal{J} && \text{Lemma 5.1.2} \\
&\iff (\mathbb{DL}_k\, h\, s, \mathbb{DL}_k\, h\, t) \in r_\alpha^{\mathbb{DL}_k(\mathcal{J},e)} && \text{def. } r_\alpha^{\mathbb{DL}_k(\mathcal{J},e)}
\end{aligned}
$$

Concept names follow similarly.

(2) $\mathbb{DL}_k(g \circ f) = (\mathbb{DL}_k\, g) \circ (\mathbb{DL}_k\, f)$ and $\mathbb{DL}_k\, id_\mathcal{I} = id_{\mathbb{DL}_k \mathcal{I}}$ equations are satisfied.

$$
\begin{aligned}
\mathbb{DL}_k(g \circ f)s &= [(g \circ f)\, a_0, \alpha_1, (g \circ f)\, a_1, ..., \alpha_j, (g \circ f)\, a_j] && \text{def. } \mathbb{DL}_k(g \circ f) \\
&= [g(f a_0), \alpha_1, g(f\, a_1), ..., \alpha_j, g(f\, a_j)] && \text{def. } \circ \\
&= \mathbb{DL}_k\, g\, [f a_0, \alpha_1, f\, a_1, ..., \alpha_j, f\, a_j] && \text{def. } \mathbb{DL}_k\, g \\
&= \mathbb{DL}_k\, g\, (\mathbb{DL}_k f s) && \text{def. } \mathbb{DL}_k\, f \\
&= (\mathbb{DL}_k\, g) \circ (\mathbb{DL}_k\, f)s
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{DL}_k\, id_\mathcal{I}\, s &= [id_\mathcal{I}\, a_0, \alpha_1, id_\mathcal{I}\, a_1, ..., \alpha_j, id_\mathcal{I}\, a_j] && \text{def. } \mathbb{DL}_k\, id_\mathcal{I} \\
&= [a_0, \alpha_1, a_1, ..., \alpha_j, a_j] = s && \text{def. } id_\mathcal{I} \\
&= id_{\mathbb{DL}_k \mathcal{I}}\, s && \text{def. } id_{\mathbb{DL}_k \mathcal{I}}
\end{aligned}
$$

□

Proposition 5.1.4. *$\varepsilon_\mathcal{I}$ is a morphism in $\mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$*

Proof. We need to show that $\varepsilon_\mathcal{I}$ is a homomorphism and that it preserves the distinguished elements. Suppose $(s,t) \in r_\alpha^{\mathbb{DL}_k(\mathcal{I},d)}$. Then $(\varepsilon_\mathcal{I} s, \varepsilon_\mathcal{I} t) \in r_\alpha^\mathcal{I}$ by the definition of interpretation. A distinguished element is represented by a singleton $[d]$ and since counit takes the last elements it clearly preserves them. The case for concept names is similar.

□

Proposition 5.1.5. *$\varepsilon : \mathbb{DL}_k \longrightarrow 1_{\mathcal{R}_*(\emptyset,\sigma_c,\sigma_r)}$ is a natural transformation.*

Proof. For arbitrary $(\mathcal{I}, d), (\mathcal{J}, e) \in \mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$, we need to show that

$$
\begin{array}{ccc}
\mathbb{DL}_k(\mathcal{I}, d) & \xrightarrow{\ \varepsilon_\mathcal{I}\ } & (\mathcal{I}, d) \\
{\scriptstyle \mathbb{DL}_k h}\downarrow & & \downarrow{\scriptstyle h} \\
\mathbb{DL}_k(\mathcal{J}, e) & \xrightarrow[\ \varepsilon_\mathcal{J}\ ]{} & (\mathcal{J}, e)
\end{array}
$$

From Proposition 5.1.4 we already know that $\varepsilon_{\mathcal{I}}$ and $\varepsilon_{\mathcal{J}}$ are morphisms. What is left to show is that the diagram commutes:

$$
\begin{aligned}
(h \circ \varepsilon_{\mathcal{I}})[a_0, \alpha_1, a_1, ..., \alpha_j, a_j] &= h\ a_j && \text{def. } \varepsilon_{\mathcal{I}} \\
&= \varepsilon_{\mathcal{J}}\ [h\ a_0, \alpha_1, h\ a_1, ..., \alpha_j, h\ a_j] && \text{def. } \varepsilon_{\mathcal{J}} \\
&= (\varepsilon_{\mathcal{J}} \circ \mathbb{DL}_k\ h)\ [a_0, \alpha_1, a_1, ..., \alpha_j, a_j] && \text{def. } \mathbb{DL}_k\ h
\end{aligned}
$$

□

Proposition 5.1.6. *The triple* $(\mathbb{DL}_k, \varepsilon, (\cdot)^*)$ *is a comonad in Kleisli form on* $\mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$

Proof. From Proposition 5.1.5 we have that ε is a natural transformation and from Proposition 5.1.3 that $\mathbb{DL}_k$ is a functor. We need to show now that the comonadic laws are satisfied and that Kleisli extension behaves as expected. Precisely, we need to prove the following properties:

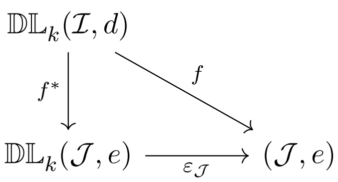
- *(A)* $\varepsilon^*_{\mathcal{I}} = id_{\mathbb{DL}_k(\mathcal{I},d)}$
- *(B)* $\varepsilon \circ f^* = f$
- *(C)* $(g \circ f^*)^* = g^* \circ f^*$
- *(D)* if h is a morphism in $\mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$ then h^* is a morphism in $\mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$

Let $s'' = [a_0, \alpha_1, a_1, ..., \alpha_{j-2}, a_{j-2}], s' = s''[\alpha_{j-1}, a_{j-1}], s = s'[\alpha_j, a_j]$. We will prove the comonad laws extensionally.

(A) We start by showing that Kleisli extension of counit yields an identity.

$$
\begin{aligned}
\varepsilon^*_{\mathcal{I}} s &= (\varepsilon^*_{\mathcal{I}} s')[\alpha_j, \varepsilon_{\mathcal{I}} s] && \text{def. } (-)^* \\
&= (\varepsilon^*_{\mathcal{I}} s'')[\alpha_{j-1}, \varepsilon_{\mathcal{I}} s', \alpha_j, \varepsilon_{\mathcal{I}} s] && \text{def. } (-)^* \\
&= [\varepsilon_{\mathcal{I}}[a_0], \alpha_1, \varepsilon_{\mathcal{I}}[a_0, \alpha_1, a_1], ..., \alpha_{j-1}, \varepsilon_{\mathcal{I}} s', \alpha_j, \varepsilon_{\mathcal{I}} s] && \text{apply inductively} \\
&= [a_0, \alpha_1, a_1, ..., \alpha_{j-1}, a_{j-1}, \alpha_j, a_j] = s && \text{def. } \varepsilon_{\mathcal{I}} \\
&= id_{\mathbb{DL}_k(\mathcal{I},d)} s
\end{aligned}
$$

(B) Let $(\mathcal{I}, d), (\mathcal{J}, e) \in \mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$ and $f : \mathbb{DL}_k(\mathcal{I}, d) \to (\mathcal{J}, e)$. Then the following diagram commutes:



$$
\begin{aligned}
(\varepsilon_{\mathcal{J}} \circ f^*)s &= \varepsilon_{\mathcal{J}}(f^* s) \\
&= \varepsilon_{\mathcal{J}}(f^*(s')[\alpha_j, f\ s)]) && \text{def. } (-)^* \\
&= f\ s && \text{def. } \varepsilon_{\mathcal{J}}
\end{aligned}
$$

(C) Let $(\mathcal{I}, d), (\mathcal{J}, e), (\mathcal{K}, k) \in \mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$ and $f : \mathbb{DL}_k(\mathcal{I}, d) \to (\mathcal{J}, e)$, $g : \mathbb{DL}_k(\mathcal{J}, e) \to (\mathcal{K}, k)$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{DL}_k(\mathcal{I}, d) & \xrightarrow{(g \circ f^*)^*} & \\ \downarrow f^* & \searrow & \\ \mathbb{DL}_k(\mathcal{J}, e) & \xrightarrow[g^*]{} & \mathbb{DL}_k(\mathcal{K}, k) \end{array}$$

$$\begin{aligned}
(g \circ f^*)^* s &= (g \circ f^*)^*(s')[\alpha_j, (g \circ f^*)\, s)] && \text{def. } (-)^* \\
&= (g \circ f^*)^*(s'')[\alpha_{j-1}, (g \circ f^*)\, s', \alpha_j, (g \circ f^*)\, s] && \text{def. } (-)^* \\
&= [(g \circ f^*)[a_0], \alpha_1, (g \circ f^*)[a_0, \alpha_1, a_1], ..., \alpha_{j-1}, (g \circ f^*)\, s', \alpha_j, (g \circ f^*)\, s] && \text{ind.} \\
&= [g(f^*[a_0]), \alpha_1, g(f^*[a_0, \alpha_1, a_1]), ..., \alpha_{j-1}, g(f^*\, s'), \alpha_j, g(f^*\, s)] \\
&= (1)
\end{aligned}$$

since $f^*[a_0] \sqsubseteq f^*[a_0, \alpha_1, a_1] \sqsubseteq ... \sqsubseteq f^* s' \sqsubseteq f^* s$, we get that

$$\begin{aligned}
(1) &= g^*(f^*\, s) \\
&= (g^* \circ f^*)\, s
\end{aligned}$$

(D) Suppose that h is a morphism in $\mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$.

$$\begin{aligned}
(s, t) \in r_\alpha^{\mathbb{DL}_k(\mathcal{I}, d)} &\implies (h\, s, h\, t) \in r_\alpha^{\mathcal{I}} && \text{h is homo.} \\
&\implies (\varepsilon_{\mathcal{I}}(h^*\, s), \varepsilon_{\mathcal{I}}(h^*\, t)) \in r_\alpha^{\mathcal{J}} && \text{by } (B) \\
&\implies (h^*\, s, h^*\, t) \in r_\alpha^{\mathbb{DL}_k(\mathcal{J}, e)} && \text{def. } r_\alpha^{\mathbb{DL}_k(\mathcal{J}, e)}
\end{aligned}$$

□

Having the $\mathcal{ALC}$-comonad defined, as the next step we introduce sufficient categorical background required to define bisimulation games in an abstract-enough way.

5.2 Tree-like structures, paths and embeddings.

A *covering relation* $\prec$ for a partial order $\leq$ is a relation satisfying $x \prec y \triangleq x \leq y \wedge x \neq y \wedge (\forall z. x \leq z \leq y \implies z = x \vee z = y)$. This is employed to define tree-like structures below, which will intuitively serve as the description of bisimulation game strategies.

Definition 5.2.1. An *ordered interpretation* $(\mathcal{I}, d, \leq)$ is a pointed interpretation $(\mathcal{I}, \mathrm{d})$ equipped with a partial order on $\Delta^{\mathcal{I}}$ such that $\uparrow(d) \triangleq \{d' \in \Delta^{\mathcal{I}} \mid d \leq d'\}$ is a tree order that satisfies the following condition (**D**) for $x, y \in \uparrow(d)$, we have $x \prec y$

iff $(x, y) \in r^{\mathcal{I}}$ for some $r \in \sigma_r$. Morphisms between ordered interpretations preserve the covering relation. We put $\mathcal{R}^D_{*k}(\mathcal{V})$ to be the category of ordered interpretation as objects with k bounding the height of the underlying tree.

We next define different kinds of embeddings, essential to characterize plays.

Definition 5.2.2. A morphism in $\mathcal{R}^D_{*k}(\mathcal{V})$ is an *embedding* if it is an injective strong homomorphism. We write $e : \mathcal{I} \rightarrowtail \mathcal{J}$ to mean that e is an embedding. Now, we define a subcategory *Paths* of $\mathcal{R}_*(\mathcal{V})$ whose objects have linear tree orders, so they comprise a single branch. We say that $e : P \rightarrowtail \mathcal{I}$ is a *path embedding* if P is a path. A morphism $f : \mathcal{I} \to \mathcal{J} \in |\mathcal{R}^D_{*k}(\mathcal{V})|$ is a *pathwise embedding* if for any path embedding $e : P \rightarrowtail \mathcal{I}$, $f \circ e$ is a path embedding.

Let $\sqsubseteq$ be the lexicographical order on sequences from $\Delta^{\mathcal{I}}$. From the construction of $\mathcal{R}^D_{*k}(\mathcal{V})$, we can extract a free functor, for which construction is justified by the following lemma:

Lemma 5.2.3. *There exists a canonical functor* $F_k\,\mathcal{I} = (\mathbb{DL}_k(\mathcal{I}, d), \sqsubseteq)$.

Proof. The proof is heavy and relies on several categorical notions that are not crucial for the paper hence we do not introduce them here; consult [10, Chapters 9 & 10.3] instead. The goal is to describe the desired functor in a way such that it yields the canonical, terminal resolution of a comonad $\mathbb{DL}_k$. First, from [8, Theorem 9.5] we know that for any $k > 0$, the Eilenberg-Moore category $EM(\mathbb{DL}_k)$ is isomorphic to $\mathcal{R}^D_{*k}(\mathcal{V})$. Having that, we can observe that there is a forgetful functor $U_k : \mathcal{R}^D_{*k}(\mathcal{V}) \to \mathcal{R}_*(\mathcal{V})$ mapping $(\mathcal{I}, d, \leq)$ to $(\mathcal{I}, d)$ which forgets the partial order. Thus, we can employ the result that follows from [8, Theorem 9.6] to infer that the functor U_k has a right adjoint F_k. The relationship between introduced categories is depicted on the diagram below, where the arrow from *Paths* to $\mathcal{R}_*(\mathcal{V})$ is the evident inclusion functor.

$$EM(\mathbb{DL}_k) \cong \mathcal{R}^D_{*k}(\mathcal{V}) \underset{U_k}{\overset{F_k}{\rightleftarrows}} \mathcal{R}_*(\mathcal{V}) \longleftarrow \textit{Paths}$$

The comonad arising from $F \dashv U$ adjunction is precisely $\mathbb{DL}_k$. □

5.3 A categorical view on games

Given a sufficient background, we can move on to the main result, namely, the characterisation of $\equiv^{\mathcal{ALC}_k}$ in the language of category theory. We start with defining what it means for a morphism in $f : \mathcal{I} \to \mathcal{J} \in |\mathcal{R}^D_{*k}(\mathcal{V})|$ to be *open*. This holds if, whenever we have a commutative square as on the LHS then there is an embedding $Q \rightarrowtail \mathcal{I}$ such that the diagram on the RHS commutes.

$$\begin{array}{ccc} P & \rightarrowtail & Q \\ \downarrow & & \downarrow \\ \mathcal{I} & \xrightarrow[f]{} & \mathcal{J} \end{array} \qquad\qquad \begin{array}{ccc} P & \rightarrowtail & Q \\ \downarrow & \swarrow & \downarrow \\ \mathcal{I} & \xrightarrow[f]{} & \mathcal{J} \end{array}$$

Finally, we can define *back-and-forth equivalence* $(\mathcal{I}, d) \leftrightarrow_k^{\mathbb{DL}} (\mathcal{J}, e)$ between objects in $\mathcal{R}_*(\mathcal{V})$, intuitively corresponding to conditions (b) and (c) from the definition of a bisimulation. This holds if there is an object R in $\mathcal{R}_{*k}^{D}(\mathcal{V})$ and a span of open pathwise embeddings such that:

$$\begin{array}{ccc} & R & \\ \swarrow & & \searrow \\ F_k(\mathcal{I}, d) & & F_k(\mathcal{J}, e) \end{array}$$

We shall now define a back-and-forth game $\mathcal{G}_k^{\Phi}(\mathcal{I}, d; \mathcal{J}, e)$ played between the interpretations $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$. Positions of the game are pairs $(s, t) \in \mathbb{DL}_k^{\Phi}(\mathcal{I}, d) \times \mathbb{DL}_k^{\Phi}(\mathcal{J}, e)$. We define a relation $W(\mathcal{I}, d; \mathcal{J}, e) \subseteq \mathbb{DL}_k^{\Phi}(\mathcal{I}, d) \times \mathbb{DL}_k^{\Phi}(\mathcal{J}, e)$ as follows. A pair (s, t) is in $W(\mathcal{I}, d; \mathcal{J}, e)$ iff for some path P, *path embeddings* $e_1 : P \rightarrowtail \mathcal{I}$ and $e_2 : P \rightarrowtail \mathcal{J}$, and $p \in P$, $s = e_1\, p$ and $t = e_2\, p$. The intention is that $W(\mathcal{I}, d; \mathcal{J}, e)$ picks out the winning positions for Duplicator. At the start of each round of the game, the position is specified by $(s, t) \in \mathbb{DL}_k^{\Phi}(\mathcal{I}, d) \times \mathbb{DL}_k^{\Phi}(\mathcal{J}, e)$. The initial position is $([d], [e])$. The round proceeds as follows. Spoiler either chooses $s' \succ s$, and Duplicator must respond with $t' \succ t$, producing the new position (s', t'); or Spoiler chooses $t'' \succ t$, and Duplicator must respond with $s'' \succ s$, producing the new position (s'', t''). Duplicator wins the round if she can respond, and the new position is in $W(\mathcal{I}, d; \mathcal{J}, e)$. We follow the same notation convention as for $\mathbb{DL}_k^{\Phi}$ with respect to extensions Φ of the game $\mathcal{G}_k^{\Phi}$. The following theorem follows from [8, Theorem 10.1].

Theorem 5.3.1. *Duplicator has a winning strategy in $\mathcal{G}_k(\mathcal{I}, d; \mathcal{J}, e)$ game if and only if $(\mathcal{I}, d) \leftrightarrow_k^{\mathbb{DL}} (\mathcal{J}, e)$.*

The above theorem with the aforementioned definitions were just slight variations of theorems and notions presented in [8]. We have accommodated them to the description logic setting and now we will glue them together with our definition of the bisimulation game from Chapter 3.

Theorem 5.3.2. *Given interpretations $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$, the $\mathcal{G}_k(\mathcal{I}, d; \mathcal{J}, e)$ game for the $\mathbb{DL}_k$ comonad is equivalent to the k-round $\mathcal{ALC}(\mathcal{V})$-bisimulation game between $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$.*

Proof. First, note that configurations and the moves are structurally the same in both games. Hence, by induction over k it suffices to show that the winning conditions coincide.

Base. Let $k = 0$ and suppose $([d], [e]) \in W(\mathcal{I}, d; \mathcal{J}, e)$. That holds iff there are path embeddings $e_1 : P \rightarrowtail \mathcal{I}$, $e_2 : P \rightarrowtail \mathcal{J}$ and $p \in P$ such that $e_1\, p = [d]$ and $e_2\, p = [e]$. By strong homomorphism property, d is in $\mathcal{V}$-harmony with p, which in turn is in $\mathcal{V}$-harmony with d, which by transitivity of $\mathcal{V}$-harmony concludes this case.

Step. Assume that the proposition holds for all $i \leq k$. We need to show that the winning conditions coincide for games of length $k + 1$. Suppose $s = s'[\alpha_s, d'], t = t'[\alpha_t, e']$ and $(s, t) \in W(\mathcal{I}, d; \mathcal{J}, e)$. That holds iff there are path embeddings $e_1 : P \rightarrowtail \mathcal{I}$, $e_2 : P \rightarrowtail \mathcal{J}$ and $p \in P$ such that $e_1\, p = s$ and $e_2\, p = t$. By definition of $W(\mathcal{I}, d; \mathcal{J}, e)$ relation, we get that $(s', t') \in W(\mathcal{I}, d; \mathcal{J}, e)$ and hence, by the induction hypothesis, s, t are a valid winning configuration in $\mathcal{ALC}$ game. It remains to show that $[\alpha_s, d']$ and $[\alpha_t, e']$ are valid moves leading to winning positions. From $e_1\, p = s$ and $e_2\, p = t$ we immediately get that $\alpha_s = \alpha_t$ and since e_1, e_2 are embeddings we have that d' is in $\mathcal{V}$-harmony with p which in turn is in $\mathcal{V}$-harmony with e', hence by transitivity of $\mathcal{V}$-harmony, we are done. □

By applying Theorem 5.3.1, Theorem 5.3.2 and Fact 3.2.2, we derive our first result on comonadic semantics for description logic games, namely:

Theorem 5.3.3. $(\mathcal{I}, d) \equiv^{\mathcal{ALC}_k} (\mathcal{J}, e) \iff (\mathcal{I}, d) \leftrightarrow_k^{\mathbb{DL}} (\mathcal{J}, e)$.

Chapter 6

Comonads for extensions of $\mathcal{ALC}$

We have defined description logic comonad in the previous chapter and in Chapter 4 we have constructed a family of game reductions that eliminate the logic extensions. By leveraging cautious categorical operations, we now combine these two and arrive at having game comonads for all considered extensions of $\mathcal{ALC}$.

6.1 A generalized framework for extensions

The approach that we undertook relies on an observation that we had based on how I-morphisms were incorporated in [8]. In our case, relative comonads serve as a tool to start within the base category where our objects live and then enrich the interpretations encoding the additional capabilities available in bisimulation games for richer logics. We do this via the already-presented reductions from Chapter 4, followed by the notion of unravelling using $\mathbb{DL}_k$ defined in Chapter 5, all established in a generalised framework using relative comonads.

Definition 6.1.1. A *vocabulary-map* δ is a triple $(\delta_i, \delta_c, \delta_r) : \mathbf{N_I} \times \mathbf{N_C} \times \mathbf{N_R} \rightarrow \mathbf{N_I} \times \mathbf{N_C} \times \mathbf{N_R}$ that maps the vocabulary $(\sigma_i, \sigma_c, \sigma_r) \longmapsto (\delta_i(\sigma_i), \delta_c(\sigma_c), \delta_r(\sigma_r))$.

Definition 6.1.2 (Reduction functor)**.** Let δ be a vocabulary map and $\mathfrak{f}$ a game reduction. A $(\mathfrak{f}, \delta)$*-reduction-functor* is a functor $J : \mathcal{R}_*(\mathcal{V}) \rightarrow \mathcal{R}_*(\delta\ \mathcal{V})$ acting $(\mathcal{I}, d) \longmapsto (\mathfrak{f}^{\mathcal{I}}\ \mathcal{I}, \mathfrak{f}^*\ d)$.

While Definition 6.1.2 is stated in a general setting, we only consider the reductions from Chapter 4. Clearly, the functors map objects to objects. When it comes to morphisms, however, we need to handle a certain delicacy. To make reasoning simpler, let us focus for a moment on $\mathcal{ALC}_{\mathsf{Self}}$. Notice that interpretations that are $\mathcal{ALC}$-homomorphic are not necessarily $\mathcal{ALC}_{\mathsf{Self}}$-homomorphic, as that would mean that self operator is expressible in bare $\mathcal{ALC}$, which we know is not the case. Consecutively, that means that homomorphic interpretations are not necessarily homomorphic after applying $\mathfrak{f}_{\mathsf{Self}}$ reduction.

To tackle this issue, we shall submerse ourselves into a particular wide subcategory, a subcategory containing all the objects of the category of interest.

Definition 6.1.3. Given $\Phi \subseteq \{\mathsf{Self}, \mathcal{I}, b, \mathcal{O}\}$, a Φ-subcategory of $\mathcal{R}_*(\mathcal{V})$ is a subcategory of $\mathcal{R}_*(\mathcal{V})$ with all objects from $\mathcal{R}_*(\mathcal{V})$ and morphisms limited to $\mathcal{ALC}\Phi$-homomorphisms.

Proof. We need to show that the Φ-subcategory of $\mathcal{R}_*(\mathcal{V})$ indeed forms a category. First, it is easy to see that we still have identity morphisms on objects. Second, $\mathcal{ALC}\Phi$-homomorphisms are closed under composition which concludes the proof. □

From now on, when considering a set of extensions Φ, we shall work in a Φ-subcategory. In this setting, the action on morphisms for reduction functors is an identity, as the very same homomorphism will work as per Theorem 4.2.1. To restrain the reader from drowning in overly verbose notation, the underlying Φ-subcategory will be taken implicitly from the context. To sum up, we obtain a family of $(\mathfrak{f}_\theta, \delta_\theta)$-reduction-functors, where $\theta \in \{\mathsf{Self}, \mathcal{I}, b, \mathcal{O}\}$ are considered logic extensions.

Definition 6.1.4. Let δ, δ' be a vocabulary-maps. We say that a functor $F : \mathcal{R}_*(\mathcal{V}) \to \mathcal{R}_*(\delta\, \mathcal{V})$ is invariant over vocabulary-maps iff for any δ' it can be lifted to $F_{\delta'} : \mathcal{R}_*(\delta'\, \mathcal{V}) \to \mathcal{R}_*(\delta\ (\delta'\, \mathcal{V}))$. We shall omit the subscript should the coercion be unambiguous.

Lemma 6.1.5. *Invariance over vocabulary maps behaves well under composition,* i.e.*, the composition of functors invariant over vocabulary maps yields a functor invariant over vocabulary maps.*

Proof. Let $F : \mathcal{R}_*(\mathcal{V}) \longrightarrow \mathcal{R}_*(\delta\, \mathcal{V})$, $G : \mathcal{R}_*(\delta\, \mathcal{V}) \longrightarrow \mathcal{R}_*(\delta'\, \mathcal{V})$ be functors invariant over vocabulary maps. We want to show that $(G \circ F) : \mathcal{R}_*(\mathcal{V}) \longrightarrow \mathcal{R}_*(\delta' \mathcal{V})$ is invariant over vocabulary maps. Let us take any vocabulary map δ''. By assumption, we can lift F, G to $F_{\delta''} : \mathcal{R}_*(\delta''\, \mathcal{V}) \longrightarrow \mathcal{R}_*((\delta \circ \delta'')\, \mathcal{V})$, $G_{\delta''} : \mathcal{R}_*((\delta \circ \delta'')\, \mathcal{V}) \longrightarrow \mathcal{R}_*((\delta' \circ \delta'')\, \mathcal{V})$. Then such composition is of the form $(G_{\delta''} \circ F_{\delta''}) : \mathcal{R}_*(\delta''\, \mathcal{V}) \longrightarrow \mathcal{R}_*((\delta' \circ \delta'')\, \mathcal{V})$ and thus $(G \circ F)$ is invariant over vocabulary maps.

$$\begin{array}{ccc} \mathcal{R}_*(\mathcal{V}) & \xrightarrow{\;G \circ F\;} & \mathcal{R}_*(\delta'\, \mathcal{V}) \\ {\scriptstyle F}\downarrow & \nearrow_{G} & \\ \mathcal{R}_*(\delta\, \mathcal{V}) & & \end{array} \qquad \begin{array}{ccc} \mathcal{R}_*(\delta''\, \mathcal{V}) & \xrightarrow{\;(G \circ F)_{\delta''}\;} & \mathcal{R}_*((\delta' \circ \delta'')\, \mathcal{V}) \\ {\scriptstyle F_{\delta''}}\downarrow & \nearrow_{G_{\delta''}} & \\ \mathcal{R}_*((\delta \circ \delta'')\, \mathcal{V}) & & \end{array}$$

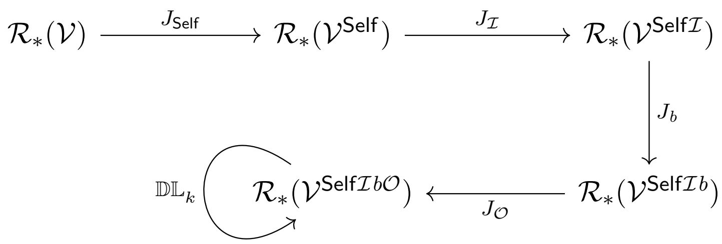
□

What we want to capture by this is that such a functor acting on $\mathcal{R}_*(\mathcal{V})$ category is natural in $\mathcal{V}$, *i.e.* does not depend on the contents of the concepts or roles. It is easy to see the following facts:

Observation 6.1.6. *$\mathbb{DL}_k$ is invariant over vocabulary-maps.*

Observation 6.1.7. *$(\mathfrak{f}_\theta, \delta_\theta)$-reduction-functors are invariant over vocabulary-maps.*

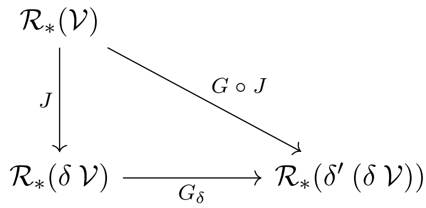
To obtain richer semantics, we shall leverage the functor composition, following the same order as defined for the game reductions in Chapter 4:



Lemma 6.1.8. *Reduction-functors are closed under composition.*

Proof. Let $J : \mathcal{R}_*(\mathcal{V}) \longrightarrow \mathcal{R}_*(\delta\mathcal{V})$ and $G : \mathcal{R}_*(\mathcal{V}) \longrightarrow \mathcal{R}_*(\delta'\mathcal{V})$ be reduction-functors. We want to show that $G \circ J$ is also a reduction-functor. Using Observation 6.1.7, we can lift G to $G : \mathcal{R}_*(\delta\,\mathcal{V}) \longrightarrow \mathcal{R}_*(\delta'\,(\delta\,\mathcal{V}))$. Let $\mathfrak{f}$, $\mathfrak{g}$ be the game reductions for J, G, respectively. Then the action on objects for $G \circ J$ is defined as follows:

$$
\begin{aligned}
G \circ J : \mathcal{R}_*(\mathcal{V}) &\longrightarrow \mathcal{R}_*(\delta'\,(\delta\,\mathcal{V})) \\
(G \circ J)\ (\mathcal{I}, d) &\longmapsto ((\mathfrak{g}^{\mathcal{I}} \circ \mathfrak{f}^{\mathcal{I}})\,\mathcal{I},\ (\mathfrak{g}^* \circ \mathfrak{f}^*)\ d).
\end{aligned}
$$



From Lemma 6.1.5, we get that the obtained composition is still invariant over vocabulary maps. ☐

6.2 Comonadic semantics for extensions

Having defined appropriate notions and tools, we now present the way to obtain game semantics for an arbitrary sublogic $\mathcal{ALC} \subseteq \mathcal{L}\Phi \subseteq \mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$ by the use of relative comonads.

Let $J_\Phi \triangleq \bigcirc_{\theta\in\Phi} J_\theta$ be a family of functors indexed by Φ where J_θ are $(\mathfrak{f}_\theta, \delta_\theta)$-reduction-functors and the operator $\bigcirc$ iterates over the extensions and composes the functors together in $(\mathsf{Self}, \mathcal{I}, b, \mathcal{O})$ order. It follows from Lemma 6.1.8 that for a fixed Φ, the functor $J_\Phi : \mathcal{R}_*(\mathcal{V}) \longrightarrow \mathcal{R}_*(\mathcal{V}^\Phi)$ is also a reduction-functor.

Proposition 6.2.1 ($\mathcal{ALC}\Phi$-comonad)**.** *The game comonad $\mathbb{DL}_k^\Phi$ is a $(\mathbb{DL}_k \circ J_\Phi)$-relative-comonad.*

Proof. We know that $J_\Phi : \mathcal{R}_*(\mathcal{V}) \longrightarrow \mathcal{R}_*(\mathcal{V}^\Phi)$ is a functor. From Proposition 5.1.6, we know that $\mathbb{DL}_k^\Phi : \mathcal{R}_*(\mathcal{V}) \longrightarrow \mathcal{R}_*(\mathcal{V})$ is a comonad on $\mathcal{R}_*(\mathcal{V})$. Applying Observation 6.1.6, we get $\mathbb{DL}^\Phi_{k\,(-)^\Phi} : \mathcal{R}_*(\mathcal{V}^\Phi) \longrightarrow \mathcal{R}_*(\mathcal{V}^\Phi)$ which is a comonad on the codomain of J_Φ. Hence, by definition, $\mathbb{DL}_k^\Phi$ is a relative comonad. □

With that, we arrive at the concluding lemma which shall guide us to the final result.

Lemma 6.2.2. *Let $k \in \mathbb{N} \cup \{\omega\}$ and let $\Phi \subseteq \{\mathsf{Self}, \mathcal{I}, b, \mathcal{O}\}$. Given pointed interpretations $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$, the $\mathcal{G}_k^\Phi(\mathcal{I}, d; \mathcal{J}, e)$ game for the $\mathbb{DL}_k^\Phi$ relative comonad is equivalent to the k-round $\mathcal{ALC}\Phi(\mathcal{V})$-bisimulation game played on $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$.*

Proof. By Theorem 4.2.1, it suffices to show that $\mathcal{G}_k^\Phi(\mathcal{I}, d; \mathcal{J}, e)$ is equivalent to $\mathcal{ALC}(\mathcal{V}^\Phi)$-bisimulation game between $(\mathfrak{f}_\Phi^\mathcal{I}\, \mathcal{I}, \mathfrak{f}_\Phi^*\, d)$ and $(\mathfrak{f}_\Phi^\mathcal{I}\, \mathcal{J}, \mathfrak{f}_\Phi^*\, e)$. Recall that the positions in the $\mathcal{G}_k^\Phi(\mathcal{I}, d; \mathcal{J}, e)$ are pairs $(s, t) \in \mathbb{DL}_k^\Phi(\mathcal{I}, d) \times \mathbb{DL}_k^\Phi(\mathcal{J}, e)$. By unfolding the definition of $\mathbb{DL}_k^\Phi$, we get that it corresponds to a product of unravelings $(\mathfrak{f}_\Phi\, \mathcal{I}, d) \times (\mathfrak{f}_\Phi\, \mathcal{J}, e)$. Hence, s and t are sequences of the form $[a_0, \alpha_1, a_1, ..., \alpha_j, a_j]$, where $\alpha_i \in \sigma_r^\Phi$ and $a_i \in \Delta^\mathcal{I} \vee a_i \in \Delta^\mathcal{J}$ for $1 \leq i \leq j$. An attentive reader can already notice that it is the same as positions in the $\mathcal{ALC}(\mathcal{V})$-bisimulation game by definition in Chapter 3. What remains to be shown is that the winning conditions coincide. Note that after applying Theorem 4.2.1 we are playing the $\mathcal{ALC}$-bisimulation game, and thus the same inductive reasoning applies as in Theorem 5.3.2 which concludes the proof.

□

For the readers that are still alive and managed to get to this point, we have finally arrived at the heart of our result. This is summarised by the following theorem, which is an immediate corollary from Fact 3.2.2, Lemma 6.2.2 and Theorem 5.3.1.

Theorem 6.2.3. *For any $k \in \mathbb{N} \cup \{\omega\}$ and a logic $\mathcal{L}\Phi$ between $\mathcal{ALC}$ and $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$, t.f.a.e.:*

- *Duplicator has the winning strategy in the k-round $\mathcal{L}\Phi(\mathcal{V})$-bisimulation-game on $(\mathcal{I}, \mathrm{d}; \mathcal{J}, \mathrm{e})$,*
- *There is an $\mathcal{L}\Phi(\mathcal{V})$-k-bisimulation $\mathcal{Z}$ between $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{Z}(\mathrm{d}, \mathrm{e})$,*
- $(\mathcal{I}, \mathrm{d}) \equiv_k^{\mathcal{L}\Phi(\mathcal{V})} (\mathcal{J}, \mathrm{e})$,
- $(\mathcal{I}, d) \leftrightarrow_k^{\mathbb{DL}^\Phi} (\mathcal{J}, e)$.

Chapter 7

Conclusions

This paper provides yet another view on bisimulation games used in the description logic setting, via the lenses of comonadic semantics, as well as another nail for the comonads hammer developed in recent years.

We have tweaked modal comonad [8] to match description logic's setting of interpretations, and devised a composable and extensible way of tackling logic extensions via reduction functors and relative monads [9]. We now shall discuss the potential directions of what can be done next.

7.1 Future Work

7.1.1 Incorporating other known DL extensions

There wo more $\mathcal{ALC}$ extensions that caught our attention, namely, counting capabilities and universal role. Following the way graded modalities were handled in [8], we believe that $\mathcal{ALCQ}$, an extension with counting capabilities, can be encoded by taking isomorphism in the Kleisli category of $\mathbb{DL}_k^\Phi$ comonad in place of $\leftrightarrow_k^{\mathbb{DL}}$ back-and-forth relation. Concerning universal role, it appears to be expressible by defining a reduction $\mathfrak{f}_U$ that adds a fresh role r_U that forms a clique. However, neither of the ideas has been carefully verified and thus that is yet to be explored.

7.1.2 Combinatorial properties

Another research direction is to investigate combinatorial properties naturally arising from the coalgebras of the resulting comonad, such as tree width for the pebbling comonad [3] or tree depth for the modal comonad [8]. A topic closely related that generalizes over parameters is the examination of $\mathbb{DL}_k^\Phi$ functor's Kan extension that should yield discrete density comonad [4].

7.1.3 Transcribing known theorems to category theory

This lies at the core of the meaning and purpose of defining comonadic semantics for model comparison games. S. Abramsky et. al has given a generalization of the framework by Arboreal categories and covers [7], and we have observed a variety of results arising from a categorical framework such as new Lovász-Type Theorems [13] or axiomatic account of Feferman-Vaught-Mostowski theorems [16]. A systematic overview of the current state of the art in applying tools from category theory in finite model theory and descriptive complexity is given in [1]. Hence, the most natural direction for the next research project would be to explore how the description logic comonad could help to generalize or simplify known theorems.

Bibliografia

[1] Samson Abramsky. *Structure and Power: an emerging landscape*. 2022.

[2] Samson Abramsky. “Whither semantics?” W: *Theor. Comput. Sci. 2020* ().

[3] Samson Abramsky, Anuj Dawar i Pengming Wang. “The pebbling comonad in Finite Model Theory”. W: *LICS 2017* (2017).

[4] Samson Abramsky, Tomás Jakl i Thomas Paine. “Discrete density comonads and graph parameters”. W: (2022).

[5] Samson Abramsky i Dan Marsden. “Comonadic semantics for guarded fragments”. W: *LICS 2021* (2021).

[6] Samson Abramsky i Dan Marsden. “Comonadic semantics for hybrid logic and bounded fragments”. W: *arXiV 2021* (2021).

[7] Samson Abramsky i Luca Reggio. “Arboreal Categories and Resources”. W: *ICALP 2021* (). Red. Nikhil Bansal, Emanuela Merelli i James Worrell. DOI: 10.48550/ARXIV.2205.05387. URL: https://arxiv.org/abs/2205.05387.

[8] Samson Abramsky i Nihil Shah. “Relating structure and power: Comonadic semantics for computational resources”. W: *J. Log. Comput. 2021* (2021).

[9] Thorsten Altenkirch, James Chapman i Tarmo Uustalu. “Monads need not be endofunctors”. W: *Log. Methods Comput. Sci. 2015* 11.1 (2015).

[10] Steve Awodey. *Category Theory*. Ebsco Publishing, 2006.

[11] Franz Baader i in. *An Introduction to Description Logic*. 2017.

[12] Ludovic Brieulle, Chan Le Duc i Pascal Vaillant. “Reasoning in the Description Logic ALC under Category Semantics”. W: *arXiv 2022* (2022).

[13] Anuj Dawar, Tomás Jakl i Luca Reggio. “Lovász-Type Theorems and Game Comonads”. W: *LICS 2021* ().

[14] Ali Rezaei Divroodi i Linh Anh Nguyen. “On bisimulations for description logics”. W: *Inf. Sci. 2015* (2015).

[15] Chan Le Duc. “Category-theoretical Semantics of the Description Logic ALC”. W: *arXiv 2021* (2021).

[16] Tomáš Jakl, Dan Marsden i Nihil Shah. *A game comonadic account of Courcelle and Feferman-Vaught-Mostowski theorems*. 2022.

[17] Saunders MacLane. *Categories for the Working Mathematician.* 1971.

[18] Robert Piro. "Model-theoretic characterisations of description logics". Prac. dokt. University of Liverpool, UK, 2012.